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Separation of variables for the A_2 Ruijsenaars model and a new integral representation for the A_2 Macdonald polynomials

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Abstract. Using the Baker–Akhiezer function technique we construct a separation of variables for the classical trigonometric three-particle Ruijsenaars model (a relativistic generalization of the Calogero–Moser–Sutherland model). In the quantum case, an integral operator M is constructed from the Askey–Wilson contour integral. The operator M transforms the eigenfunctions of the commuting Hamiltonians (the Macdonald polynomials for the root sytem A_2) into the factorized form $S(y_1)S(y_2)$ where $S(y)$ is a Laurent polynomial of one variable expressed in terms of the ${}_3\phi_2(y)$ basic hypergeometric series. The inversion of M produces a new integral representation for the A_2 Macdonald polynomials. We also present some results and conjectures for the general n-particle case.

1. Introduction

The separation of variables (SoV) is an approach to quantum integrable systems which can be formulated briefly as follows (for a more detailed discussion see the survey [1]).

Given a quantum-mechanical system of n degrees of freedom possessing n commuting Hamiltonians

$$[H_j, H_k] = 0 \qquad j, k = 1, 2, \ldots, n \tag{1.1}$$

one tries to find an operator M that transforms any common eigenvector P_λ of the Hamiltonians

$$H_j P_\lambda = h_j P_\lambda \tag{1.2}$$

labelled by the quantum numbers $\lambda = \{\lambda_1, \ldots, \lambda_n\}$ into the product

$$M : P_\lambda \to \prod_{j=1}^{n} S_{\lambda;j}(y_j) \tag{1.3}$$

of functions $S_{\lambda;j}(y_j)$ each of one variable. The original multi-dimensional eigenvalue problem (1.2) is transformed respectively into a set of simpler one-dimensional spectral problems (separated equations)

$$\mathcal{D}_j\left(y_j, \frac{\partial}{\partial y_j}; h_1, \ldots, h_n\right) S_{\lambda;j}(y_j) = 0 \tag{1.4}$$

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where the $\mathcal{D}_j$ are usually some differential or finite-difference operators in the variable y_j depending on the spectral parameters h_k. In the context of classical Hamiltonian mechanics the above construction corresponds precisely to the standard definition of SoV in the Hamilton–Jacobi equation.

The advent of the inverse scattering method gave new life to SoV, providing it with the interpretation of the separated coordinates y_j (in the classical case) as the poles of the Baker–Akhiezer (BA) function (properly the normalized eigenvector of the corresponding Lax matrix). The unsolved question is, however, how to choose a correct normalization of the BA function to obtain SoV for a given Lax matrix. Nevertheless, as a heuristic recipe, the above idea has proved to be quite efficient and allowed one to find SoV for a few new classes of classical integrable systems. In particular, SoV was found for the systems arising from the r-matrices satisfying the classical Yang–Baxter equation in case of A_{n-1} (sl_n) Lie algebra. In the cases $n = 2$ and $n = 3$ the construction of SoV has been successfully transferred to the quantum case (see [1] and references therein).

Pursuing the goal of extending the applicability of the BA function recipe, in our previous paper [2] we studied the A_2 Calogero–Sutherland (CS) model, which does not fall into the previously studied cases since it posesses a dynamical (non-numeric) r-matrix [3]. In the quantum case, our construction of SoV produced a new integral representation for the eigenfunctions of the A_2 CS Hamiltonians (known as Jack polynomials) in terms of ${}_3F_2$ hypergeometric polynomials.

In the present paper we generalize the results of [2] to the three-particle Ruijsenaars model [4], which is a relativistic analogue of the CS model. The corresponding eigenfunctions (Macdonald polynomials [5, 6]) are q-analogues of Jack polynomials. It comes as no surprise that the corresponding separated functions are Laurent polynomials expressed in terms of ${}_3\phi_2$ basic hypergeometric series. We also present some results and conjectures for the general n-particle problem, for instance, we connect the A_{n-1} type basic hypergeometric separation polynomials $S_\lambda(y)$ to a terminated case of the ϕ_D type q-Lauricella function of $n - 1$ variables.

The paper is organized as follows. In section 2 we describe the classical Ruijsenaars model and, using the BA function technique, construct a SoV. Although the results of this section are not used directly in what follows, they provide a useful background for the subsequent treatment of the quantum case. In section 3 the standard facts concerning the quantum Ruijsenaars model and Macdonald polynomials are brought together. In section 4, after introducing the quantum Hamiltonians and Macdonald polynomials, we describe the integral operator M performing a SoV and formulate the main theorem whose proof takes the rest of section 4 and part of section 5. The main part of the proof is contained in section 4 where the properties of the operator M are studied, whereas in section 5 the results concerning the separated equation (a certain third-order q-difference equation and its nth order generalization), as well as its polynomial solutions, are collected. The main technical tool allowing us to study the operator M is the famous Askey–Wilson integral identity (A.15).

Generally, SoV is aimed at simplifying the multidimensional spectral problem by reducing it to a series of one-dimensional problems. In the case of the Calogero–Sutherland and Ruijsenaars models, however, the spectrum and eigenfunctions are well known and are studied by independent means. The main benefit of SoV in application to these models lies rather in the production of new relations between special functions. In particular, inverting the operator M one obtains a new integral representation for the A_2 Macdonald polynomials in terms of the ${}_3\phi_2$ basic hypergeometric functions, which is done at the end of section 4. In section 6 we discuss the results thus obtained and the possibility of their generalization to the A_n, $n > 2$ case. Two appendices, A and B, respectively contain a collection of necessary formulae from q-analysis and some auxiliary results concerning the operator M.

2. The classical Ruijsenaars model

In the spirit of q-analysis, we prefer to use exponentiated canonical coordinates and momenta.

Definition 1. The variables (X_j, x_j) $j = 1, \ldots, n$ on a $2n$-dimensional symplectic manifold form a *Weyl canonical system* if they possess the Poisson brackets

$$\{X_j, X_k\} = \{x_j, x_k\} = 0 \quad \{X_j, x_k\} = -\mathrm{i} X_j x_k \delta_{jk} \qquad j, k = 1, \ldots, n \tag{2.1}$$

or, equivalently, the symplectic form ω is expressed as $\omega = \mathrm{i} \sum_j \mathrm{d} \ln X_j \wedge \mathrm{d} \ln x_j = \mathrm{d}(\mathrm{i} \sum_j \ln X_j \mathrm{d} \ln x_j)$.

The n-particle (A_{n-1}) trigonometric Ruijsenaars model [4] is formulated in terms of the Weyl canonical system (T_j, t_j) where $|t_j| = 1$, $T_j \in \mathbb{R}$ $(j = 1, 2, \ldots, n)$. The Hamiltonians H_i are defined as

$$H_i = \sum_{\substack{J \subset \{1,\ldots,n\} \\ |J| = i}} \left(\prod_{\substack{j \in J \\ k \in \{1,\ldots,n\} \setminus J}} v_{jk} \right) \left(\prod_{j \in J} T_j \right) \qquad i = 1, \ldots, n \tag{2.2}$$

where

$$v_{jk} = \frac{\ell^{-\frac{1}{2}} t_j - \ell^{\frac{1}{2}} t_k}{t_j - t_k} \qquad \ell \in (1, \infty) . \tag{2.3}$$

Proposition 1 ([4, 7]). The Hamiltonians H_j Poisson commute:

$$\{H_j, H_k\} = 0 \qquad j, k = 1, \ldots, n. \tag{2.4}$$

Define the Lax matrix (L operator) by the formula

$$L(u) = D(u) E(u) \tag{2.5}$$

where

$$D_{jk} = \frac{(\ell - 1)(1 - \ell^n u)}{2\ell^{(n+1)/2}(1 - u)} \left(\prod_{i \neq j} v_{ji} \right) T_j \delta_{jk} \tag{2.6}$$

$$E_{jk} = \frac{1 + \ell^n u}{1 - \ell^n u} - \frac{t_j + \ell t_k}{t_j - \ell t_k}. \tag{2.7}$$

Proposition 2 ([4]). The characteristic polynomial of the matrix $L(u)$, equation (2.5), generates the Hamiltonians (2.2)

$$(-1)^n \ell^{n(n-1)/2} (1 - \ell^n u)(1 - u)^n \det(z - L(u))$$

$$= \sum_{k=0}^{n} (-1)^k \ell^{[(n-1)/2]k} (1 - \ell^k u)(1 - u)^k (1 - \ell^n u)^{n-k} H_{n-k} z^k \tag{2.8}$$

where we assume $H_0 \equiv 1$.

In the three-particle (A_2) case which we consider hereafter we have, respectively,

$$H_1 = v_{12}v_{13}T_1 + v_{21}v_{23}T_2 + v_{31}v_{32}T_3 \tag{2.9a}$$

$$H_2 = v_{13}v_{23}T_1T_2 + v_{12}v_{32}T_1T_3 + v_{21}v_{31}T_2T_3 \tag{2.9b}$$

$$H_3 = T_1T_2T_3 \tag{2.9c}$$

$$D = \frac{(\ell-1)(1-\ell^3u)}{2\ell^2(1-u)}\,\mathrm{diag}\,\{v_{12}v_{13}T_1,\, v_{21}v_{23}T_2,\, v_{31}v_{32}T_3\} \tag{2.10}$$

$$E_{jk} = \frac{1+\ell^3u}{1-\ell^3u} - \frac{t_j+\ell t_k}{t_j-\ell t_k} \tag{2.11}$$

and

$$\ell^3(1-u)^2\det(z-L(u)) = z^3\ell^3(1-u)^2 - z^2\ell^2(1-u)(1-\ell^2u)H_1 \\ +z\ell(1-\ell u)(1-\ell^3u)H_2 - (1-\ell^3u)^2H_3. \tag{2.12}$$

To find a SoV for the Ruijsenaars system we use the recipe discussed in section 1 and choose for the separated coordinates y_j the poles upon u of the Baker–Akhiezer function $\psi(u)$ (an eigenvector of $L(u)$) normalized by the condition that its third component $\psi_3(u)$ be a constant. The canonically conjugated (in the Weyl sense) variables Y_j are chosen as the eigenvalues of $L(y_j)$. For a detailed discussion of the BA function recipe see [1], though the construction described below is quite self-contained.

Define two functions $A_1(u)$ and $A_2(u)$ by the formulae

$$A_k(u) := L_{kk} - \frac{L_{3k}L_{k,3-k}}{L_{3,3-k}} = T_k\alpha_k(u) \qquad k=1,2 \tag{2.13}$$

$$\alpha_k(u) := \frac{(1-\ell^3u)(\ell t_3u - t_{3-k})(t_k-\ell t_3)}{\ell(1-u)(\ell^2t_3u-t_{3-k})(\ell t_k - t_3)} \qquad k=1,2\,. \tag{2.14}$$

The separated variables y_j are defined by the formula

$$A_1(y) = A_2(y). \tag{2.15}$$

It is easy to see that (2.15) has three solutions, one of which ($y=\ell^{-3}$) we ignore since it is a constant. The remaining two roots we denote y_1 and y_2. From the easily verified invariance of $\alpha_1(u)/\alpha_2(u)$ under the transformation $u\mapsto u^{-1}t_1t_2t_3^{-2}\ell^{-\frac{3}{2}}$ it follows that

$$y_1y_2 = \frac{t_1t_2}{t_3^2\ell^3}. \tag{2.16}$$

The conjugated variables Y_j are defined as

$$Y_j = A_1(y_j) = A_2(y_j) \qquad j=1,2. \tag{2.17}$$

Equivalently, the four variables Y_1, Y_2, y_1, y_2 are defined through four equations

$$Y_j = T_k\alpha_k(y_j) \qquad j,k\in\{1,2\}. \tag{2.18}$$

Theorem 1. The variables Y_j, y_j satisfy the separated equations

$$Y_j^3\ell^3(1-y_j)^2 - Y_j^2\ell^2(1-y_j)(1-\ell^2y_j)H_1 + Y_j\ell(1-\ell y_j)(1-\ell^3y_j)H_2 \\ -(1-\ell^3y_j)^2H_3 = 0 \qquad j=1,2 \tag{2.19}$$

which, by virtue of (2.12), imply that $\det(Y_j - L(y_j)) = 0$.

Proof. Substitute in (2.19) the expressions (2.9) for H_j and split the left-hand side of (2.19) into two terms

$$T_3 Z_1 + Y_j Z_2 = 0 \tag{2.20}$$

where

$$Z_1 = -(1-y_j)(1-\ell^2 y_j)\ell^2 v_{31} v_{32} Y_j^2 + (1-\ell y_j)(1-\ell^3 y_j)\ell(v_{12}v_{32}T_1 Y_j + v_{21}v_{31}T_2 Y_j) \\ -(1-\ell^3 y_j)^2 T_1 T_2 \tag{2.21a}$$

$$Z_2 = (1-y_j)^2 \ell^3 Y_j^2 - (1-y_j)(1-\ell^2 y_j)\ell^2(v_{12}v_{13}T_1 Y_j + v_{21}v_{23}T_2 Y_j) \\ +(1-\ell y_j)(1-\ell^3 y_j)v_{13}v_{23}T_1 T_2. \tag{2.21b}$$

To prove (2.19) it is sufficient to show that $Z_1 = Z_2 = 0$. Replacing Y_j in (2.21) by $T_1\alpha_1(y_j)$ or $T_2\alpha_2(y_j)$ in such a way that the factor T_1T_2 could be cancelled from $Z_{1,2}$ we obtain that $Z_{1,2} = 0$ follows from two algebraic identities for $\alpha_{1,2}$

$$-(1-y)(1-\ell^2 y)\ell^2 v_{31}v_{32}\alpha_1(y)\alpha_2(y) \\ +(1-\ell y)(1-\ell^3 y)\ell\big(v_{12}v_{32}\alpha_2(y) + v_{21}v_{31}\alpha_1(y)\big) - (1-\ell^3 y)^2 = 0 \tag{2.22a}$$

$$(1-y)^2\ell^3\alpha_1(y)\alpha_2(y) - (1-y)(1-\ell^2 y)\ell^2\big(v_{12}v_{13}\alpha_2(y) + v_{21}v_{23}\alpha_1(y)\big) \\ +(1-\ell y)(1-\ell^3 y)\ell v_{13}v_{23} = 0 \tag{2.22b}$$

which are verified directly. □

The third pair of separated variables is defined as

$$x := t_3 \qquad X := T_1 T_2 T_3 \tag{2.23}$$

the corresponding separated equation being

$$X - H_3 = 0. \tag{2.24}$$

Theorem 2. The variables $(X, Y_1, Y_2; x, y_1, y_2)$ form a Weyl canonical system in the sense of definition 1.

Proof. Let us introduce new variables

$$t_+ = t_1^{1/2} t_2^{1/2} t_3^{-1} \qquad t_- = t_1^{1/2} t_2^{-1/2} \tag{2.25a}$$

$$T_+ = T_1 T_2 \qquad T_- = T_1 T_2^{-1} \tag{2.25b}$$

and also

$$y_+ = y_1^{1/2} y_2^{1/2} \qquad y_- = y_1^{1/2} y_2^{-1/2} \tag{2.26a}$$

$$Y_+ = Y_1 Y_2 \qquad Y_- = Y_1 Y_2^{-1}. \tag{2.26b}$$

Obviously, $(X, T_-, T_+; x, t_-, t_+)$ is also a Weyl canonical system. Note that

$$y_+ = t_+ \ell^{-\frac{3}{2}} \tag{2.27}$$

because of (2.16). Note also that from (2.14) it follows that $Y_\pm$, $y_\pm$ depend only on $T_\pm$, $t_\pm$ and do not contain X, x.

It remains for us to show that the transformation from $(T_-, T_+; t_-, t_+)$ to $(Y_-, Y_+; y_-, y_+)$ is canonical, that is $(Y_-, Y_+; y_-, y_+)$ is again a Weyl canonical system. To this end, it suffices to construct the generating function $F(Y_+, y_-; t_+, t_-)$ of the canonical transformation such that [8]

$$\mathrm{i}\ln T_\pm = t_\pm \frac{\partial F}{\partial t_\pm} \qquad \mathrm{i}\ln Y_- = -y_- \frac{\partial F}{\partial y_-} \qquad \mathrm{i}\ln y_+ = Y_+ \frac{\partial F}{\partial Y_+}. \tag{2.28}$$

and $\mathrm{d}(F - \mathrm{i}\ln Y_+ \ln y_+) = \mathrm{i}(\ln T_- \mathrm{d}\ln t_- + \ln T_+ \mathrm{d}\ln t_+) - \mathrm{i}(\ln Y_- \mathrm{d}\ln y_- + \ln Y_+ \mathrm{d}\ln y_+)$. Recalling the definition of the Euler dilogarithm [9]

$$\mathrm{Li}_2(z) := -\int_0^z \frac{\mathrm{d}t}{t}\ln(1-t) = \sum_{k=1}^{\infty} \frac{z^k}{k^2} \tag{2.29}$$

and introducing the notation

$$\mathcal{L}(\nu; x, y) := \mathrm{Li}_2(\nu x y) + \mathrm{Li}_2(\nu x y^{-1}) + \mathrm{Li}_2(\nu x^{-1} y) + \mathrm{Li}_2(\nu x^{-1} y^{-1}) \tag{2.30}$$

we define $F := \mathrm{i}\ln Y_+ \ln(\ell^{-\frac{3}{2}} t_+) + \widetilde{F}$:

$$\widetilde{F} := \mathrm{i}\big(\mathcal{L}(\ell^{-\frac{1}{2}}; y_-, t_-) + \mathcal{L}(\ell^{-1}; t_+, t_-) - \mathcal{L}(\ell^{-\frac{3}{2}}; t_+, y_-) - \mathrm{Li}_2(t_-^2) - \mathrm{Li}_2(t_-^{-2})\big). \tag{2.31}$$

It is a matter of direct calculation to verify, using equations (2.18) and (2.27), that F satisfies (2.28). □

The identities (2.19) and (2.24) and canonicity of the variables $(X, Y_1, Y_2; x, y_1, y_2)$ established above provide, by definition [1], a SoV for the A_2 Ruijsenaars system.

3. Quantization

Here we collect the standard facts concerning the quantum n-particle (A_{n-1}) Ruijsenaars model [4, 7] and the corresponding Macdonald polynomials [5, 6].

Throughout the paper $\mathbb{Z}$ stands for the set of integers, the notation $\mathbb{Z}_{\geqslant 0}$ and $\mathbb{Z}_{\leqslant 0}$ being self-evident.

The quantum Ruijsenaars model is described in terms of the multiplication and shift operators, t_j and T_j $(j = 1, \ldots, n)$ respectively, acting on the functions of t_j

$$(t_j f)(\boldsymbol{t}) := t_j f(\boldsymbol{t}) \qquad (T_j f)(\boldsymbol{t}) := f(\ldots, q t_j, \ldots) \tag{3.1}$$

(we do not make distinction between variables and operators t_j). Here q is the quantum deformation parameter related to the Planck constant $\hbar > 0$ as

$$q = \mathrm{e}^{-\hbar} \qquad q \in (0, 1). \tag{3.2}$$

The operators T_j, t_j satisfy the Weyl commutation relations

$$[T_j, T_k] = [t_j, t_k] = 0 \qquad T_j t_k = \begin{cases} q t_k T_j & j = k \\ t_k T_j & j \neq k \end{cases} \tag{3.3}$$

which produce the Poisson brackets (2.1) in the classical limit $\hbar \to 0$ by the standard correspondence rule $[,] = -\mathrm{i}\hbar\{,\} + \mathrm{O}(\hbar^2)$.

The commuting quantum Hamiltonians H_j

$$[H_j, H_k] = 0 \qquad j, k = 1, \ldots, n \tag{3.4}$$

are given by the same formulae (2.2) as in the classical case with the fixed operator ordering (T_j to the right). We assume that

$$\ell = q^{-g} = \mathrm{e}^{g\hbar} \qquad g > 0 \quad \ell \in (1, \infty) \tag{3.5}$$

(note that both in the classical and non-relativistic limits $\overline{h} \to 0$, $q = \mathrm{e}^{-\overline{h}} \to 1$ but in the classical limit $g \to \infty$, $\ell =$ constant whereas in the non-relativistic limit $g =$ constant, $\ell \to 1$).

The operators H_k leave invariant the space $\mathrm{Sym}(t_1, \dots, t_n)$ of symmetric Laurent polynomials in variables t_j. A basis in $\mathrm{Sym}(t_1, \dots, t_n)$ is given by the monomial symmetric functions m_λ labelled by the sequences $\lambda = \{\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n\}$ of integers $\lambda_j \in \mathbb{Z}$ (dominant weights) and expressed as $m_\lambda = \sum t_1^{\nu_1} \cdots t_n^{\nu_n}$ where the sum is taken over all distinct permutations ν of λ.

Denote $|\lambda| \equiv \sum_{j=1}^n \lambda_j$. The dominant order on the dominant weights λ is defined as

$$\lambda' \preceq \lambda \quad \Longleftrightarrow \quad \left\{ |\lambda'| = |\lambda|\,;\ \sum_{j=k}^{n} \lambda'_j \leqslant \sum_{j=k}^{n} \lambda_j,\ k = 2, \dots, n \right\}. \tag{3.6}$$

The Macdonald polynomials $P_\lambda^{(\ell;q)} \in \mathrm{Sym}(t_1, \dots, t_n)$ are uniquely defined as joint eigenvectors of H_k in $\mathrm{Sym}(t_1, \dots, t_n)$

$$H_k P_\lambda^{(\ell;q)} = h_k P_\lambda^{(\ell;q)} \tag{3.7}$$

labelled by the dominant weight λ and normalized by the condition

$$P_\lambda^{(\ell;q)} = \sum_{\lambda' \preceq \lambda} \kappa_{\lambda'} m_{\lambda'} \qquad \kappa_\lambda = 1. \tag{3.8}$$

The corresponding eigenvalues h_k are

$$h_k = \sum_{j_1 < \cdots < j_k} \mu_{j_1} \cdots \mu_{j_k} \qquad \mu_j = q^{\lambda_j} \ell^{(n+1)/2 - j}. \tag{3.9}$$

Note that our parameters ℓ and t used in [5, 6] relate as $\ell = t^{-1}$.

The polynomials $P_\lambda^{(\ell;q)}$ are orthogonal

$$\frac{1}{(2\pi\mathrm{i})^n} \oint_{|t_1|=1} \frac{\mathrm{d}t_1}{t_1} \cdots \oint_{|t_n|=1} \frac{\mathrm{d}t_n}{t_n} \bar{P}_\lambda^{(\ell;q)}(\boldsymbol{t}) P_{\lambda'}^{(\ell;q)}(\boldsymbol{t}) \Delta(\boldsymbol{t}) = 0 \qquad \lambda \neq \lambda' \tag{3.10}$$

with respect to the weight

$$\Delta(t_1, \dots, t_n) = \prod_{j \neq k} \frac{(t_j t_k^{-1}; q)_\infty}{(\ell^{-1} t_j t_k^{-1}; q)_\infty} \tag{3.11}$$

(see equation (A.2) for the notation).

In the limit $\overline{h} \to 0$, $g =$ constant the appropriate linear combinations of H_k produce the Hamiltonians of the non-relativistic Calogero–Sutherland model, and the Macdonald polynomials go over into the Jack polynomials, see [2].

In the present paper we consider only the simplest non-trivial case $n = 3$.

With the Hamiltonians H_k being given by (2.9), equations (3.9) respectively produce

$$h_1 = \ell q^{\lambda_1} + q^{\lambda_2} + \ell^{-1} q^{\lambda_3} \qquad h_2 = \ell q^{\lambda_1+\lambda_2} + q^{\lambda_1+\lambda_3} + \ell^{-1} q^{\lambda_2+\lambda_3} \qquad h_3 = q^{|\lambda|} \tag{3.12}$$

for their eigenvalues labelled by the ordered triplets $\{\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3\} \in \mathbb{Z}^3$.

For instance

$$m_{000} = 1 \qquad m_{001} = t_1 + t_2 + t_3 \qquad m_{011} = t_1 t_2 + t_1 t_3 + t_2 t_3$$

$$m_{002} = t_1^2 + t_2^2 + t_3^2 \qquad m_{111} = t_1 t_2 t_3$$

$$m_{012} = t_1 t_2^2 + t_1^2 t_2 + t_1 t_3^2 + t_1^2 t_3 + t_2 t_3^2 + t_2^2 t_3 \qquad m_{112} = t_1^2 t_2 t_3 + t_1 t_2^2 t_3 + t_1 t_2 t_3^2$$

$$m_{022} = t_1^2 t_2^2 + t_1^2 t_3^2 + t_2^2 t_3^2 \qquad m_{003} = t_1^3 + t_2^3 + t_3^3$$

$$P^{(\ell;q)}_{000} = m_{000} \qquad P^{(\ell;q)}_{001} = m_{001} \qquad P^{(\ell;q)}_{011} = m_{011}$$

$$P^{(\ell;q)}_{002} = m_{002} + \frac{(1-\ell)(1+q)}{q-\ell} m_{011}$$

$$P^{(\ell;q)}_{111} = m_{111} \qquad P^{(\ell;q)}_{012} = m_{012} + \frac{(1-\ell)(q(2+\ell)+1+2\ell)}{q-\ell^2} m_{111}$$

$$P^{(\ell;q)}_{112} = m_{112} \qquad P^{(\ell;q)}_{022} = m_{022} + \frac{(1-\ell)(1+q)}{q-\ell} m_{112}$$

$$P^{(\ell;q)}_{003} = m_{003} + \frac{(1-\ell)(1+q+q^2)}{q^2-\ell} m_{012} + \frac{(1-\ell)^2(1+q)(1+q+q^2)}{(q-\ell)(q^2-\ell)} m_{111}.$$

4. The operator M

We are now going to describe the integral operator M (1.3) producing the SoV. Generally speaking, the kernel $\mathcal{M}$ of M should depend on six variables: $\mathcal{M}(x, y_1, y_2 \mid t_1, t_2, t_3)$. However, by analogy with the classical case (section 2) and the non-relativistic limit [2], it is natural to assume that $\mathcal{M}$ contains two δ-functions corresponding to the constraints $x = t_3$ (2.23) and, respectively, (2.16). There remains thus only one integration in M. Again by analogy with the previously studied cases, the kernel $\mathcal{M}$ is most conveniently described in terms of the variables $t_\pm$ (2.25*a*) and $y_\pm$ (2.26*a*).

So, let us introduce the operator M

$$M : \Psi(t_1, t_2, t_3) \to \Phi(x, y_1, y_2)$$
$$= \frac{1}{2\pi \mathrm{i}} \int_{\Gamma^{t_+,y_-}_{g,2g}} \frac{\mathrm{d}t_-}{t_-} \mathcal{M}\big((y_1y_2)^{\frac{1}{2}}, (y_1/y_2)^{\frac{1}{2}} \,\big|\, t_-\big) \Psi\big(\ell^{\frac{3}{2}} x (y_1y_2)^{\frac{1}{2}} t_-, \ell^{\frac{3}{2}} x (y_1y_2)^{\frac{1}{2}} t_-^{-1}, x\big) \tag{4.1}$$

with the kernel

$$\mathcal{M}(y_+, y_- \mid t_-) = \frac{(1-q)(q;q)^2_\infty (t_-^2, t_-^{-2}; q)_\infty \, \mathcal{L}_q\Big(\ell^{-\frac{3}{2}}; y_-, y_+\ell^{\frac{3}{2}}\Big)}{2B_q(g, 2g)\, \mathcal{L}_q\Big(\ell^{-\frac{1}{2}}; y_-, t_-\Big)\, \mathcal{L}_q\Big(\ell^{-1}; t_-, y_+\ell^{\frac{3}{2}}\Big)} \tag{4.2}$$

where the notation (A.7) and (B.2) is used. For the definition of the cycle $\Gamma^{t_+,y_-}_{g,2g}$ which depends on g, $y_{1,2}$ see equations (B.4) and (A.16).

Remark. In the classical limit, as $q \to 1$, $\ell =$ constant, using (A.20) and $\ln \mathcal{L}_q(\nu; x, y) \sim -\overline{h}^{-1}\mathcal{L}(\nu; x, y)$ one obtains that the asymptotics $\ln \mathcal{M} \sim -\mathrm{i}\overline{h}^{-1}\widetilde{F}$ of the kernel $\mathcal{M}$ is determined by the regular part $\widetilde{F}$ (2.31) of the generating function of the canonical transformation producing classical SoV. As for the non-relativistic limit, $\overline{h} \to 0$, $g =$ constant, the easiest way to reproduce the results of [2] is to compare the action of the operator M and its non-relativistic analogue on polynomials, see theorem 4.

Now we are in a position to formulate our main result.

Theorem 3. The operator M, equation (4.1), transforms any A_2 Macdonald polynomial $P^{(\ell;q)}_\lambda(t_1, t_2, t_3)$ into the product

$$M : P^{(\ell;q)}_\lambda(t_1, t_2, t_3) \to c_\lambda x^{|\lambda|} S^{(\ell;q)}_\lambda(y_1) S^{(\ell;q)}_\lambda(y_2) \tag{4.3}$$

of functions of one variable only, where the Laurent polynomials $S^{(\ell;q)}_{\lambda_1\lambda_2\lambda_3}$

$$S^{(\ell;q)}_{\lambda}(y) = \sum_{k=\lambda_1}^{\lambda_3} \chi^{(\ell;q)}_{\lambda,k} y^k \tag{4.4}$$

are expressed in terms of the basic hypergeometric series (A.9)

$$S^{(\ell;q)}_{\lambda}(y) = y^{\lambda_1}(y;q)_{1-3g}\, {}_3\phi_2 \left[\begin{matrix} \ell^3 q^{1-\lambda_{31}}, \ell^2 q^{1-\lambda_{21}}, \ell q \\ \ell^2 q^{1-\lambda_{31}}, \ell q^{1-\lambda_{21}} \end{matrix} ; q, y \right] \tag{4.5}$$

where $\lambda_{jk} \equiv \lambda_j - \lambda_k$. The coefficients $\chi^{(\ell;q)}_{\lambda,k}$ are given by

$$\chi^{(\ell;q)}_{\lambda,k} = (q\ell^3)^{k-\lambda_1} \frac{(q^{-1}\ell^{-3};q)_{k-\lambda_1}}{(q;q)_{k-\lambda_1}}\, {}_4\phi_3 \left[\begin{matrix} q^{\lambda_1-k}, \ell^3 q^{1-\lambda_{31}}, \ell^2 q^{1-\lambda_{21}}, \ell q \\ \ell^3 q^{\lambda_1-k+2}, \ell^2 q^{1-\lambda_{31}}, \ell q^{1-\lambda_{21}} \end{matrix} ; q, q \right]. \tag{4.6}$$

The normalization coefficient c_λ equals

$$c_\lambda = \ell^{4\lambda_1-\lambda_2} \frac{(\ell^{-2};q)_{\lambda_{31}}(\ell^{-2};q)_{\lambda_{32}}(\ell^{-1};q)_{\lambda_{21}}}{(\ell^{-3};q)_{\lambda_{31}}(\ell^{-1};q)_{\lambda_{32}}(\ell^{-2};q)_{\lambda_{21}}}. \tag{4.7}$$

The proof of the above result will occupy the rest of this section and a part of the next one. Our proof parallels the similar one for the non-relativistic Calogero–Sutherland model [2].

We begin with proving the factorization (4.3) of $MP^{(\ell;q)}_\lambda$. The first step is to show that the image $MP^{(\ell;q)}_\lambda$ satisfies certain q-difference equations in x, y_1, y_2. Let us introduce the operators Y_j ($j = 1, 2$) acting on functions of y_k as (cf (3.1))

$$(Y_j f)(\boldsymbol{y}) = f(\ldots, qy_j, \ldots). \tag{4.8}$$

Using $y_\pm = (y_1 y_2^{\pm 1})^{1/2}$, equation (2.26*a*), one can also write

$$(Y_1 f)(y_+, y_-) = f(q^{\frac{1}{2}}y_+, q^{\frac{1}{2}}y_-) \qquad (Y_2 f)(y_+, y_-) = f(q^{\frac{1}{2}}y_+, q^{-\frac{1}{2}}y_-). \tag{4.9}$$

Similarly

$$(T_1 f)(t_+, t_-) = f(q^{\frac{1}{2}}t_+, q^{\frac{1}{2}}t_-) \qquad (T_2 f)(t_+, t_-) = f(q^{\frac{1}{2}}t_+, q^{-\frac{1}{2}}t_-). \tag{4.10}$$

We define also the operator X as $X(f)(x) = f(qx)$.

Let us introduce the operator expression $\mathcal{D}$

$$\begin{aligned} \mathcal{D}(u, z; \mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3) :=\ & (1-qu)(1-q^2u)\ell^3 z^3 - (1-qu)(1-q^2\ell^2 u)\ell^2 z^2 \mathcal{H}_1 \\ & + (1-q\ell u)(1-q^2\ell^3 u)\ell z \mathcal{H}_2 - (1-q\ell^3 u)(1-q^2\ell^3 u)\mathcal{H}_3 \end{aligned} \tag{4.11}$$

which can be considered as a quantum generalization of the characteristic polynomial (2.12). The ordering is important in (4.11) since we are going to replace the parameters u, z, $\mathcal{H}_j$ by non-commuting operators.

Proposition 3. The operator M (4.1) satisfies the equations

$$XM - MH_3 = 0 \tag{4.12}$$

$$\mathcal{D}(y_j, Y_j; MH_1, MH_2, MH_3) = 0 \qquad j = 1, 2 \tag{4.13}$$

where $H_{1,2,3}$ are the quantum Hamiltonians (2.9).

Proof. Although the equality (4.12) can easily be derived from the fact that M respects the constraint $x = t_3$, or directly from (4.1), we shall instead proceed in a more methodical fashion, allowing us to prove both (4.12) and (4.13) in the same way. Let us first rewrite the operator identities (4.12) and (4.13) for M as algebraic identities for the kernel $\mathcal{M}$ (4.2).

We define the Lagrange adjoint Hamiltonians H_k^* as

$$H_1^* = T_1^{-1} v_{12} v_{13} + T_2^{-1} v_{21} v_{23} + T_3^{-1} v_{31} v_{32} \tag{4.14a}$$

$$H_2^* = T_1^{-1} T_2^{-1} v_{13} v_{23} + T_1^{-1} T_3^{-1} v_{12} v_{32} + T_2^{-1} T_3^{-1} v_{21} v_{31} \tag{4.14b}$$

$$H_3^* = T_1^{-1} T_2^{-1} T_3^{-1} \tag{4.14c}$$

$$\frac{1}{2\pi \mathrm{i}} \oint \frac{\mathrm{d}t}{t}\, f(t)(Hg)(t) = \frac{1}{2\pi \mathrm{i}} \oint \frac{\mathrm{d}t}{t}\, (H^* f)(t) g(t). \tag{4.15}$$

In particular, $T_j^* = T_j^{-1}$. Considering M in (4.12) and (4.13) as an integral operator, we can use integration by parts and switch H_k to the kernel $\mathcal{M}$ replacing them by H_k^* according to (4.15) which results in the q-difference equations for $\mathcal{M}$:

$$(X - H_3^*)\mathcal{M} = 0 \tag{4.16}$$

$$\mathcal{D}(y_j, Y_j; H_1^*, H_2^*, H_3^*)\mathcal{M} = 0 \qquad j = 1, 2. \tag{4.17}$$

While equation (4.16) is obvious, equation (4.17) needs more consideration. Note that, by virtue of (4.9) and (4.10), the action of $\mathcal{D}$ on $\mathcal{M}(y_+, y_- \mid t_-)$ is well defined. Note also that equations (4.16) and (4.17) are respectively the quantum counterparts of the classical separated equations (2.24) and (2.19).

The next step is to notice that the kernel $\mathcal{M}$ (4.2) satisfies the four first-order q-difference equations

$$Y_j T_k \mathcal{M} = \check{\alpha}_k(y_j)\mathcal{M} \qquad j, k \in \{1, 2\} \tag{4.18}$$

where (cf the classical equation (2.14))

$$\check{\alpha}_k(y) = \frac{(1 - q\ell^3 y)(t_k - \ell t_3)(\ell t_3 y - t_{3-k})(q t_k - t_{3-k})}{\ell(1 - y)(q\ell t_k - t_3)(q\ell^2 t_3 y - t_{3-k})(t_k - t_{3-k})} \qquad k = 1, 2 \tag{4.19}$$

which are verified directly from (4.2) using relations (A.4). Note that equation (4.18) is the quantum counterpart of (2.17)–(2.18).

Remark. It is easy to verify that the system (4.18) is holonomic, that is the operators $\check{\alpha}_k(y_j)^{-1} Y_j T_k$ commute, provided y_j and t_k are bound by (2.16).

We now proceed with the derivation of the third-order q-difference relations in y_j, equation (4.17), for $\mathcal{M}$ from the first-order relations (4.18). The proof parallels that of theorem 1 for the classical case. Let us write down the equations (4.17) explicitly:

$$\begin{aligned}\big[(1 - q y_j)(1 - q^2 y_j)\ell^3 Y_j^3 - (1 - q y_j)(1 - q^2\ell^2 y_j)\ell^2 Y_j^2 H_1^* + (1 - q\ell y_j)(1 - q^2\ell^3 y_j)\ell Y_j H_2^* \\ - (1 - q\ell^3 y_j)(1 - q^2\ell^3 y_j) H_3^*\big]\mathcal{M} = 0\end{aligned} \tag{4.20}$$

then substitute in (4.20) the expressions (4.14) for H_j^* and split the left-hand side of (4.20) into two terms

$$T_3^{-1} \check{Z}_1 + Y_j \check{Z}_2 = 0 \tag{4.21}$$

where

$$\check{Z}_1 = \big[-(1-qy_j)(1-q^2\ell^2y_j)\ell^2Y_j^2v_{31}v_{32} - (1-q\ell^3y_j)(1-q^2\ell^3y_j)T_1^{-1}T_2^{-1} + (1-q\ell y_j)(1-q^2\ell^3y_j)\ell Y_j(T_1^{-1}v_{12}v_{32} + T_2^{-1}v_{21}v_{31})\big]\mathcal{M} \tag{4.22a}$$

$$\check{Z}_2 = \big[(1-y_j)(1-qy_j)\ell^3Y_j^2 + (1-\ell y_j)(1-q\ell^3y_j)\ell T_1^{-1}T_2^{-1}v_{13}v_{23} - (1-y_j)(1-q\ell^2y_j)\ell^2Y_j(T_1^{-1}v_{12}v_{13} + T_2^{-1}v_{21}v_{23})\big]\mathcal{M}. \tag{4.22b}$$

Introducing the notation

$$\check{\alpha}_{12}(y) \equiv \check{\alpha}_1(qy)\big|_{t_2:=qt_2}\check{\alpha}_2(y) = \check{\alpha}_2(qy)\big|_{t_1:=qt_1}\check{\alpha}_1(y) \tag{4.23}$$

$$\check{v}_{jk} = \frac{\ell^{-\frac{1}{2}}t_j - q\ell^{\frac{1}{2}}t_k}{t_j - qt_k} \tag{4.24}$$

and noting that

$$T_kv_{jk} = \check{v}_{jk}T_k \qquad v_{jk}T_j = T_j\check{v}_{jk} \tag{4.25}$$

it is easy to verify the algebraic identities for $\check{\alpha}_{1,2}$

$$(1-qy)(1-q^2\ell^2y)\ell^2\check{v}_{31}\check{v}_{32}\check{\alpha}_{12}(y) - (1-q\ell^3y)(1-q^2\ell^3y) + (1-q\ell y)(1-q^2\ell^3y)\ell\big(\check{v}_{12}\check{v}_{32}\check{\alpha}_2(y) + \check{v}_{21}\check{v}_{31}\check{\alpha}_1(y)\big) = 0 \tag{4.26a}$$

$$(1-y)(1-qy)\ell^3\check{\alpha}_{12}(y) + (1-\ell y)(1-q\ell^3y)\ell v_{13}v_{23} - (1-y)(1-q\ell^2y)\ell^2\big(\check{v}_{12}v_{13}\check{\alpha}_2(y) + \check{v}_{21}v_{23}\check{\alpha}_1(y)\big) = 0. \tag{4.26b}$$

Now we insert $T_kT_k^{-1}$ in appropriate places in (4.22) in such a way that $T_1^{-1}T_2^{-1}$ can be carried out to the left of $[\cdots]$. Then we push the products YT to the right using (4.25) until they hit $\mathcal{M}$, so that (4.18) can be applied. The equalities $\check{Z}_1 = 0$, $\check{Z}_2 = 0$ and therefore (4.20) and (4.13) then follow immediately from (4.26). □

Proposition 4. The function $\big(MP_\lambda^{(\ell;q)}\big)(x, y_1, y_2)$ satisfies the q-difference equations (separated equations)

$$(X - h_3)MP_\lambda^{(\ell;q)} = 0 \tag{4.27}$$

$$\mathcal{D}(y_j, Y_j; h_1, h_2, h_3)MP_\lambda^{(\ell;q)} = 0 \qquad j = 1, 2. \tag{4.28}$$

Proof. Apply the operator expressions (4.12) and (4.13) to the function $P_\lambda^{(\ell;q)}$. Using the operator ordering convention and the fact that Macdonald polynomials $P_\lambda^{(\ell;q)}$ are the eigenfunctions of the Hamiltonians H_j (3.7) one replaces H_j by h_j. Since h_j are just numbers, the operator M can then be applied directly to $P_\lambda^{(\ell;q)}$ which results in (4.27) and (4.28). □

In order to derive the factorization (4.3) of $MP_\lambda^{(\ell;q)}$ we need more specific information about how M acts on the symmetric polynomials from $\mathrm{Sym}(t_1, t_2, t_3)$. Note that solutions to (4.18), as to any q-difference equations, are defined only up to a factor invariant under q-shifts (quasi-constant). Our choice (4.2) of the kernel $\mathcal{M}$ corresponds to a particular choice of the quasi-constant which is crucial for the results given below.

Since the kernel $\mathcal{M}$ (4.2) is a particular case, equation (B.8), of the kernel $\mathcal{M}_{\alpha\beta}$, equation (B.7), we can make use of the results obtained for $\mathcal{M}_{\alpha\beta}$ in appendix B.

Let us define a few polynomial spaces. Let $\mathrm{Sym}(t_1, t_2, t_3)$ be the space of Laurent polynomials symmetric w.r.t. permutations of the three variables t_1, t_2, t_3. A basis in $\mathrm{Sym}(t_1, t_2, t_3)$ is given by m_λ or $P_\lambda^{(\ell;q)}$. Let $\mathrm{Sym}(t_1, t_2; t_3)$ be the space of Laurent polynomials of the same three variables, symmetric only w.r.t. $t_1 \leftrightarrow t_2$. Obviously, $\mathrm{Sym}(t_1, t_2; t_3) \supset \mathrm{Sym}(t_1, t_2, t_3)$. Though the Macdonald polynomials belong to $\mathrm{Sym}(t_1, t_2, t_3)$ it is convenient to define M on a larger space $\mathrm{Sym}(t_1, t_2; t_3)$.

Let $\mathrm{Ref}(t_-; t_+; t_3)$ be the space of Laurent polynomials in $t_\pm$, t_3 which are reflexive in t_- (invariant w.r.t. $t_- \to t_-^{-1}$) and even in $t_\pm$ (invariant w.r.t. $(t_-, t_+) \to (-t_-, -t_+)$. Note that the change of variables $(t_1, t_2, t_3) \to (t_-, t_+, t_3)$, see equation (2.25*a*), provides an isomorphism $\mathrm{Sym}(t_1, t_2; t_3) \simeq \mathrm{Ref}(t_-; t_+; t_3)$.

The spaces $\mathrm{Sym}(y_1, y_2; x) \simeq \mathrm{Ref}(y_-; y_+; x)$ are similarly defined.

Proposition 5.

$$M : \mathrm{Sym}(t_1, t_2; t_3) \to \mathrm{Sym}(y_1, y_2; x).$$

In particular, the image of a Macdonald polynomial $P_\lambda^{(\ell;q)} \in \mathrm{Sym}(t_1, t_2, t_3)$ also lies in $\mathrm{Sym}(y_1, y_2; x)$.

Proof. Proposition 13 from appendix B implies that $M : \mathrm{Ref}(t_-; t_+; t_3) \to \mathrm{Ref}(y_-; y_+; x)$. Using the isomorphisms $\mathrm{Sym}(t_1, t_2; t_3) \simeq \mathrm{Ref}(t_-; t_+; t_3)$ and $\mathrm{Sym}(y_1, y_2; x) \simeq \mathrm{Ref}(y_-; y_+; x)$ we conclude the proof. □

Now everything is ready for us to prove the main statement of theorem 3.

Proposition 6. The operator M transforms any Macdonald polynomial $P_\lambda^{(\ell;q)}$ into the product (4.3).

Proof. We have already established that $MP_\lambda^{(\ell;q)}$ is a Laurent polynomial (proposition 5) satisfying the q-difference equations (4.27) and (4.28). The factorization (4.3) follows from the fact that x^{h_3} and $S_\lambda^{(\ell;q)}(y)$ are respectively the unique, up to a constant factor, Laurent-polynomial solutions, of the q-difference equations $(X - h_3)f(x) = 0$ and $\mathcal{D}(y, Y; h_1, h_2, h_3)f(y) = 0$. The first statement is obvious, as for the second one, see proposition 12. □

Although for theorem 3 we have used only the polynomiality of $MP_\lambda^{(\ell;q)}$, in fact, the action of M on $\mathrm{Sym}(t_1, t_2; t_3)$ can be described in much more detail. Namely, taking equation (B.13) from appendix B, making the substitutions (B.8) and performing the changes of variables (2.25*a*) and (2.26*a*) one obtains the following result.

Theorem 4. Consider the basis in $\mathrm{Sym}(t_1, t_2; t_3)$

$$p_{jk\nu} := t_3^{j-2k} t_1^k t_2^k (\ell^{-1} t_1 t_3^{-1}, \ell^{-1} t_2 t_3^{-1}; q)_\nu \qquad j, k \in \mathbb{Z} \quad \nu \in \mathbb{Z}_{\geqslant 0} \tag{4.29}$$

and in $\mathrm{Sym}(y_1, y_2; x)$

$$\widetilde{p}_{jk\nu} := x^j y_1^k y_2^k (y_1, y_2; q)_\nu \qquad j, k \in \mathbb{Z} \quad \nu \in \mathbb{Z}_{\geqslant 0} \tag{4.30}$$

respectively. The operator M acts on $p_{jk\nu}$ as follows:

$$M : p_{jk\nu} \to \ell^{3k} \frac{(\ell^{-2}; q)_\nu}{(\ell^{-3}; q)_\nu} \widetilde{p}_{jk\nu}. \tag{4.31}$$

Postponing to section 5 the proof of equations (4.5) and (4.6), we can now prove the final statement of theorem 3.

Proposition 7. The normalization coefficient c_λ in (4.3) is given by (4.7).

Proof. In this case, it is convenient to make use of the isomorphisms described above and to think of M as acting from $\mathrm{Ref}(t_-;t_+;t_3)$ into $\mathrm{Ref}(y_-;y_+;x)$. Comparing the asymptotics of the monomial symmetric functions m_λ

$$m_{\lambda_1\lambda_2\lambda_3} \sim t_-^{\lambda_3-\lambda_1} t_+^{\lambda_3+\lambda_1} t_3^{\lambda_1+\lambda_2+\lambda_3} \qquad t_- \to \infty$$

and of the polynomial $p_{jk\nu}$ (4.29)

$$\begin{aligned} p_{jk\nu} &\equiv t_3^j t_+^{2k} (\ell^{-1} t_+ t_-, \ell^{-1} t_+ t_-^{-1})_\nu \\ &\sim (-1)^\nu q^{\nu(\nu-1)/2} \ell^{-\nu} t_3^j t_+^{2k+\nu} t_-^\nu \qquad t_- \to \infty \end{aligned}$$

we conclude that the transition matrix between the bases m_λ and $p_{jk\nu}$ is triangular

$$m_\lambda = (-1)^{\lambda_{31}} q^{-\lambda_{31}(\lambda_{31}-1)/2} \ell^{\lambda_{31}} p_{|\lambda|,\lambda_1,\lambda_{31}} + \sum_{\nu<\lambda_{31}} \sum_{j,k} (\cdots) p_{jk\nu}. \tag{4.32}$$

Given the mutual triangularity, equation (3.8), of the bases $P_\lambda^{(\ell;q)}$ and m_λ, it means that the expansion of $P_\lambda^{(\ell;q)}$ in $p_{jk\nu}$ has the same structure as (4.32). Then using equation (4.31) and the asymptotics of $\widetilde{p}_{jk\nu}$, equation (4.30),

$$\widetilde{p}_{jk\nu} \equiv x^j y_+^{2k} (y_+ y_-, y_+ y_-^{-1})_\nu \sim (-1)^\nu q^{\nu(\nu-1)/2} x^j y_+^{2k+\nu} y_-^\nu \qquad y_- \to \infty$$

we obtain

$$\begin{aligned} M P_\lambda^{(\ell;q)} &= (-1)^{\lambda_{31}} q^{-\lambda_{31}(\lambda_{31}-1)/2} \ell^{2\lambda_1+\lambda_3} \frac{(\ell^{-2};q)_{\lambda_{31}}}{(\ell^{-3};q)_{\lambda_{31}}} \widetilde{p}_{|\lambda|,\lambda_1,\lambda_{31}} + \cdots \\ &\sim \ell^{2\lambda_1+\lambda_3} \frac{(\ell^{-2};q)_{\lambda_{31}}}{(\ell^{-3};q)_{\lambda_{31}}} x^{|\lambda|} y_+^{\lambda_3+\lambda_1} y_-^{\lambda_{31}} \qquad y_- \to \infty. \end{aligned} \tag{4.33}$$

On the other hand, equation (4.4) implies that

$$c_\lambda\, x^{|\lambda|}\, S_\lambda^{(\ell;q)}(y_+ y_-)\, S_\lambda^{(\ell;q)}(y_+ y_-^{-1}) \sim c_\lambda\, \chi_{\lambda,\lambda_1}^{(\ell;q)}\, \chi_{\lambda,\lambda_3}^{(\ell;q)}\, x^{|\lambda|} y_+^{\lambda_3+\lambda_1} y_-^{\lambda_3-\lambda_1}$$

whence

$$c_\lambda\, \chi_{\lambda,\lambda_1}^{(\ell;q)}\, \chi_{\lambda,\lambda_3}^{(\ell;q)} = \frac{(\ell^{-2};q)_{\lambda_3-\lambda_1}}{(\ell^{-3};q)_{\lambda_3-\lambda_1}} \ell^{\lambda_3+2\lambda_1}. \tag{4.34}$$

It only remains for us to use equations (5.14) proved in section 5, and obtain equation (4.7). □

Compared with [2] our equation (4.7) for the normalization coefficients c_λ is new, and its non-relativistic analogue

$$c_\lambda = \frac{(2g)_{\lambda_{31}}(2g)_{\lambda_{32}}(g)_{\lambda_{21}}}{(3g)_{\lambda_{31}}(g)_{\lambda_{32}}(2g)_{\lambda_{21}}} \qquad (\alpha)_k \equiv \alpha(\alpha+1)\cdots(\alpha+k-1)$$

fills the gap in the description given in [2] of the integral representation for Jack polynomials analogous to (4.38).

We conclude this section with a list of results concerning the inverse operator M^{-1}. All the preparatory work having been done in appendix B, it only remains for us to use the correspondence (B.8) between $M_{\alpha\beta}$ and M.

From (B.20) and (B.21) it follows that M^{-1} is an integral operator

$$\begin{aligned} &M^{-1} : \Phi(x, y_1, y_2) \to \Psi(t_1, t_2, t_3) \\ &\quad = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{-g,3g}^{t_+,t_-}} \frac{\mathrm{d}y_-}{y_-} \widetilde{\mathcal{M}}\left(\frac{(t_1t_2)^{\frac12}}{t_3}, \left(\frac{t_1}{t_2}\right)^{\frac12} \,\middle|\, y_- \right) \Phi\left(t_3, \frac{\ell^{-\frac32}(t_1t_2)^{\frac12} y_-}{t_3}, \frac{\ell^{-\frac32}(t_1t_2)^{\frac12}}{t_3 y_-} \right) \end{aligned} \tag{4.35}$$

with the kernel

$$\widetilde{\mathcal{M}}(t_+, t_- \mid y_-) = \frac{(1-q)(q;q)_\infty^2 (y_-^2, y_-^{-2}; q)_\infty \, \mathcal{L}_q\left(\ell^{-1}; t_-, t_+\right)}{2B_q(-g, 3g)\, \mathcal{L}_q\left(\ell^{\frac{1}{2}}; y_-, t_-\right) \mathcal{L}_q\left(\ell^{-\frac{3}{2}}; y_-, t_+\right)}. \tag{4.36}$$

Reversing equation (4.31) one obtains the formula for the action of M^{-1} on the basis $\widetilde{p}_{jk\nu}$

$$M^{-1} : \widetilde{p}_{jk\nu} \to \ell^{-3k} \frac{(\ell^{-3}; q)_\nu}{(\ell^{-2}; q)_\nu} p_{jk\nu}. \tag{4.37}$$

Reversing equation (4.3) provides a new integral representation of A_2 Macdonald polynomials in terms of the Laurent polynomials $S_\lambda^{(\ell;q)}(y)$, equation (4.5),

$$M^{-1} : c_\lambda x^{|\lambda|} S_\lambda^{(\ell;q)}(y_1) S_\lambda^{(\ell;q)}(y_2) \to P_\lambda^{(\ell;q)}(t_1, t_2, t_3). \tag{4.38}$$

Finally, from propositions 14 and 15 it follows that for positive integer g the operator M^{-1} turns into a q-difference operator of order g:

$$M^{-1} : \Phi(x, y_1, y_2) \to \sum_{k=1}^{g} \xi_k\left(\frac{(t_1 t_2)^{\frac{1}{2}}}{t_3}, \left(\frac{t_1}{t_2}\right)^{\frac{1}{2}}\right) \Phi\left(t_3, q^{g+k}\frac{t_1}{t_3}, q^{2g-k}\frac{t_2}{t_3}\right) \tag{4.39}$$

where $\xi_k(r, s)$ is given by (B.15). The result is not surprising in view of the similar result for the non-relativistic case [2] where M^{-1} becomes a differential operator of order g for $g \in \mathbb{Z}_{\geqslant 0}$. In [2] this result was derived using a representation of M^{-1} in terms of the fractional differential operator. In the relativistic case it is also possible to relate M^{-1} with a sort of fractional q-difference operator. We intend to tackle this subject in a separate paper.

5. The separated equation

In this section are collected the results concerning the Laurent polynomials $S_\lambda^{(\ell;q)}(y)$ and the corresponding q-difference equations. Since all the results are easy to generalize from $n = 3$ to arbitrary n, we give them in the most general form.

Conjecture 1. The correct generalization of equation (4.5) for $S_\lambda^{(\ell;q)}(y)$ for any n is given by

$$S_\lambda^{(\ell;q)}(y) = y^{\lambda_1} (y;q)_{1-ng}\, {}_n\phi_{n-1}\left[\begin{matrix} a_1, \ldots, a_n \\ b_1, \ldots, b_{n-1} \end{matrix}; q, y\right] \tag{5.1}$$

where

$$a_j = \ell^{n-j+1} q^{\lambda_1 - \lambda_{n-j+1} + 1} \qquad b_j = a_j \ell^{-1}. \tag{5.2}$$

Proposition 8. $S_\lambda^{(\ell;q)}(y)$ is a Laurent polynomial in y of the form

$$S_\lambda^{(\ell;q)}(y) = \sum_{k=\lambda_1}^{\lambda_n} \chi_{\lambda,k}^{(\ell;q)} y^k. \tag{5.3}$$

Proof. Observe, first, that if $a = bq^\nu$ for some positive integer ν then

$$\frac{(a;q)_k}{(b;q)_k} = \frac{(bq^k;q)_\nu}{(b;q)_\nu} \tag{5.4}$$

is a polynomial in q^k of degree ν whose coefficients are rational functions in b and q. As a consequence, if $a_{j+1} = b_j q^{\nu_j}$ then

$$P_N(q^k) \equiv \frac{(a_2;q)_k \cdots (a_n;q)_k}{(b_1;q)_k \cdots (b_{n-1};q)_k} = \frac{(b_1 q^k;q)_{\nu_1} \cdots (b_{n-1} q^k;q)_{\nu_{n-1}}}{(b_1;q)_{\nu_1} \cdots (b_{n-1};q)_{\nu_{n-1}}} \tag{5.5}$$

is a polynomial in q^k of degree $N = \nu_1 + \cdots + \nu_{n-1}$.

In our case, $\nu_j = \lambda_{n-j+1} - \lambda_{n-j}$, $N = \lambda_n - \lambda_1$ by virtue of (5.2), and from (5.1) and (A.9) one obtains

$${}_n\phi_{n-1}\left[\begin{matrix} a_1, \ldots, a_n \\ b_1, \ldots, b_{n-1}\end{matrix}; q, y\right] = \sum_{k=0}^{\infty} \frac{(a_1;q)_k}{(q;q)_k}\, y^k P_N(q^k) \tag{5.6}$$

where $P_N(q^k)$ is given by (5.5). It now remains for us to apply the following lemma.

Lemma 1. Let $P_N(y)$ be a polynomial in y of degree $\leqslant N$. Then

$$\sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k}\, y^k P_N(q^k) = Q_N(y) \frac{(aq^N y;q)_\infty}{(y;q)_\infty} \tag{5.7}$$

where $Q_N(y)$ is a polynomial in y of degree $\leqslant N$.

Proof. It is sufficient to consider the polynomials $P_N(q^k) = (q^{k-\nu+1};q)_\nu$ for $\nu = 0, 1, \ldots, N$ forming a basis in the polynomial ring. Then

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k}\, y^k (q^{k-\nu+1};q)_\nu &= \sum_{k=\nu}^{\infty} [\cdots] = \sum_{k=\nu}^{\infty} \frac{(a;q)_k}{(q;q)_{k-\nu}}\, y^k \\ &= \sum_{k=0}^{\infty} \frac{(a;q)_{k+\nu}}{(q;q)_k}\, y^{k+\nu} = (a;q)_\nu y^\nu \sum_{k=0}^{\infty} \frac{(aq^\nu;q)_k}{(q;q)_k}\, y^k. \end{aligned} \tag{5.8}$$

Using equation (A.11) and the identity $(aq^\nu;q)_\infty = (aq^\nu;q)_{N-\nu}(aq^N;q)_\infty$ one finally obtains expression (5.7) where $Q_N(y) = (a;q)_\nu y^\nu (aq^\nu y;q)_{N-\nu}$. □

Applying the above lemma to the case of the polynomial $P_N(q^k)$ given by (5.6) and $a = a_1 = \ell^n q^{\lambda_1 - \lambda_n + 1}$ we finally obtain that $y^{-\lambda_1} S_\lambda^{(\ell;q)}(y)$ is a polynomial of degree $\leqslant \lambda_n - \lambda_1$. □

Proposition 9. The coefficients $\chi_{\lambda,k}^{(\ell;q)}$ in the expansion (5.3) are given by

$$\chi_{\lambda,k}^{(\ell;q)} = \left(q\ell^n\right)^{k-\lambda_1} \frac{(q^{-1}\ell^{-n};q)_{k-\lambda_1}}{(q;q)_{k-\lambda_1}}\, {}_{n+1}\phi_n\left[\begin{matrix} q^{\lambda_1-k}, a_1, \ldots, a_n \\ q^{\lambda_1-k+2}\ell^n, b_1, \ldots, b_{n-1}\end{matrix}; q, q\right]. \tag{5.9}$$

In particular, for $n = 3$, equation (5.9) produces (4.6).

Proof. We know already that $S_\lambda^{(\ell;q)}(y)$ is a Laurent polynomial and is thus defined for any $y \in \mathbb{C} \setminus \{0, \infty\}$. Suppose for a while that $|y| < 1$. Then both factors $(y;q)_{1-ng}$ and ${}_n\phi_{n-1}$ in (5.1) are given by the convergent series (A.11) and (A.9), respectively. Multiplying the two power series in y we observe that the coefficients at y^k is expressed in terms of ${}_{n+1}\phi_n$ series:

$$S_\lambda^{(\ell;q)}(y) = y^{\lambda_1} \sum_{k=0}^{\infty} \left(q\ell^n\right)^k \frac{(q^{-1}\ell^{-N};q)_k}{(q;q)_k}\, {}_{n+1}\phi_n\left[\begin{matrix} q^{-k}, a_1, \ldots, a_n \\ q^{-k+2}\ell^n, b_1, \ldots, b_{n-1}\end{matrix}; q, q\right] y^k. \tag{5.10}$$

In fact, the sum in (5.10) is finite: $\sum_{k=0}^{\lambda_{n1}}$. To see this, use [10, equation (1.9.11)]: let $\nu, k_1, \ldots, k_n \in \mathbb{Z}_{\geqslant 0}$, then

$$
{}_{n+1}\phi_n \left[\begin{matrix} q^{-\nu}, b_1 q^{k_1}, \ldots, b_n q^{k_n} \\ b_1, \ldots, b_n \end{matrix} ; q, q \right] = 0 \tag{5.11}
$$

for $\nu > k_1 + \cdots + k_n$. Substituting

$$
\begin{aligned}
&\nu = k \qquad b_n = q^{2-k}\ell^n \qquad k_n = \lambda_1 - \lambda_n + k - 1 \\
&b_j = \ell^{n-j} q^{\lambda_1 - \lambda_{n-j+1}+1} \qquad k_j = \lambda_{n-j+1} - \lambda_{n-j} \qquad j = 1, \ldots, n-1\,,
\end{aligned}
$$

we obtain that

$$
{}_{n+1}\phi_n \left[\begin{matrix} q^{-k}, a_1, \ldots, a_n \\ q^{2-k}\ell^n, b_1, \ldots, b_{n-1} \end{matrix} ; q, q \right] = 0
$$

for $k \geqslant \lambda_n - \lambda_1 + 1$, hence the sum in (5.10) is finite: $\sum_{k=0}^{\lambda_{n1}}$. The coefficient at y^k in (5.10) produces equation (5.9). □

For the sake of reference we present a short list of polynomials $S_\lambda^{(\ell;q)}(y)$ in the $n = 3$ case:

$$
S_{000}^{(\ell;q)} = 1 \qquad S_{001}^{(\ell;q)} = 1 + \frac{\ell^2 y}{\ell + 1} \qquad S_{011}^{(\ell;q)} = 1 + \ell\,(\ell+1)\,y
$$

$$
S_{002}^{(\ell;q)} = 1 + \frac{\ell^2(q\ell + \ell - q - 1)y}{\ell^2 - q} + \frac{(\ell - q)\ell^4 y^2}{(\ell^2 - q)(\ell + 1)}
$$

$$
S_{012}^{(\ell;q)} = 1 + \frac{(\ell^3 + \ell^2 q + \ell^2 - \ell q - \ell - q)\ell y}{\ell^2 - q} + \ell^3 y^2
$$

$$
S_{022}^{(\ell;q)} = 1 + \frac{(\ell^2 - 1)(q + 1)\ell y}{\ell - q} + \frac{(\ell^2 - q)(\ell + 1)\ell^2 y^2}{\ell - q}
$$

$$
S_{003}^{(\ell;q)} = 1 + \frac{(1 + q + q^2)(\ell - 1)\ell^2 y}{\ell^2 - q^2} + \frac{(1 + q + q^2)(\ell - 1)\ell^4 y^2}{(\ell + q)(\ell^2 - q)} + \frac{(\ell - q^2)\ell^6 y^3}{(\ell + 1)(\ell + q)(\ell^2 - q)}
$$

$$
\begin{aligned}
S_{013}^{(\ell;q)} = 1 &+ \frac{(\ell^3 + q^2\ell^2 + \ell^2 q + \ell^2 - q^2\ell - \ell q - \ell - q^2)\ell y}{\ell^2 - q^2} \\
&+ \frac{(q\ell^3 + \ell^3 + \ell^2 q^2 + \ell^2 q + \ell^2 - q^3\ell - \ell q^2 - \ell q - q^3 - q^2)(\ell - 1)\ell^3 y^2}{(\ell^2 - q)(\ell^2 - q^2)} \\
&+ \frac{(\ell - q)\ell^5 y^3}{\ell^2 - q}
\end{aligned}
$$

$$
\begin{aligned}
S_{023}^{(\ell;q)} = 1 &+ \frac{(q\ell^3 + \ell^3 + \ell^2 q^2 + \ell^2 q + \ell^2 - q^3\ell - q^2\ell - \ell q - q^3 - q^2)(\ell - 1)\ell y}{(\ell - q)^2(\ell + q)} \\
&+ \frac{(\ell^3 + \ell^2 q^2 + \ell^2 q + \ell^2 - q^2\ell - \ell q - \ell - q^2)(\ell^2 - q)\ell^2 y^2}{(\ell - q)^2(\ell + q)} \\
&+ \frac{(\ell^2 - q)\ell^4 y^3}{\ell - q}
\end{aligned}
$$

$$S^{(\ell;q)}_{033} = 1 + \frac{(1+q+q^2)(\ell^2-1)\ell y}{\ell - q^2} + \frac{(1+q+q^2)(\ell^2-1)(\ell^2-q)\ell^2 y^2}{(\ell-q)(\ell-q^2)} + \frac{(\ell+1)(\ell+q)(\ell^2-q)\ell^3 y^3}{\ell - q^2}\,.$$

Remark. It easy to give simpler expressions for some $\chi^{(\ell;q)}_{\lambda,k}$ such as

$$\chi^{(\ell;q)}_{\lambda,\lambda_1} = 1 \tag{5.12}$$

and

$$\chi^{(\ell;q)}_{\lambda,\lambda_n} = \ell^{|\lambda|-n\lambda_1} \prod_{j=1}^{n-1} \frac{(\ell^{-j};q)_{\lambda_j-\lambda_1}(\ell^{-j};q)_{\lambda_n-\lambda_{n-j}}}{(\ell^{-j};q)_{\lambda_{j+1}-\lambda_1}(\ell^{-j};q)_{\lambda_n-\lambda_{n-j+1}}}. \tag{5.13}$$

which for $n=3$ produce

$$\chi^{(\ell;q)}_{\lambda,\lambda_1} = 1 \qquad \chi^{(\ell;q)}_{\lambda,\lambda_3} = \ell^{-2\lambda_1+\lambda_2+\lambda_3} \frac{(\ell^{-1};q)_{\lambda_{32}}(\ell^{-2};q)_{\lambda_{21}}}{(\ell^{-2};q)_{\lambda_{32}}(\ell^{-1};q)_{\lambda_{21}}}. \tag{5.14}$$

Equation (5.12) is obvious. To obtain equation (5.13), use the summation formula [10, equation (1.9.10)]:

$${}_{n+1}\phi_n\left[\begin{matrix} q^{-\nu}, \beta, \beta_1 q^{k_1}, \ldots, \beta_{n-1}q^{k_{n-1}} \\ \beta q, \beta_1, \ldots, \beta_{n-1}\end{matrix}; q, q\right] = \frac{(q;q)_\nu(\beta_1/\beta;q)_{k_1}\cdots(\beta_{n-1}/\beta;q)_{k_{n-1}}}{(\beta q;q)_\nu(\beta_1;q)_{k_1}\cdots(\beta_{n-1};q)_{k_{n-1}}}\,\beta^\nu \tag{5.15}$$

where $\nu, k_1, \ldots, k_{n-1} \in \mathbb{Z}_{\geqslant 0}$ and $\nu \geqslant k_1 + \cdots + k_{n-1}$. Substituting

$$\nu = \lambda_{n1} \qquad \beta = \ell^n q^{1-\lambda_{n1}} \equiv a_1$$

$$\beta_j = \ell^j q^{1-\lambda_{j+1}+\lambda_1} \equiv b_{n-j} \qquad k_j = \lambda_{j+1} - \lambda_j \qquad j = 1, \ldots, n-1\,,$$

we obtain, after a series of equivalent transformations (see [10, appendix I]), expression (5.13).

Remark. There is also a simple formula for $S^{(\ell;q)}_\lambda(\ell^{-n})$:

$$S^{(\ell;q)}_\lambda(\ell^{-n}) = \ell^{-n\lambda_1}(\ell^{-n};q)_{\lambda_{n1}} \prod_{j=1}^{n-1} \frac{(\ell^{-j};q)_{\lambda_j-\lambda_1}}{(\ell^{-j};q)_{\lambda_{j+1}-\lambda_1}} \tag{5.16}$$

or, for $n=3$,

$$S^{(\ell;q)}_\lambda(\ell^{-3}) = \ell^{-3\lambda_1} \frac{(\ell^{-2};q)_{\lambda_{21}}(\ell^{-3};q)_{\lambda_{31}}}{(\ell^{-1};q)_{\lambda_{21}}(\ell^{-2};q)_{\lambda_{31}}}, \tag{5.17}$$

which are proved in a way similar to (5.13) using [10, equation (1.9.9)].

The polynomials $S^{(\ell;q)}_\lambda(y)$ can also be expressed in terms of the q-Lauricella function (A.12).

Proposition 10. The following equalities hold:

$$S^{(\ell;q)}_\lambda(y) = y^{\lambda_1} \frac{(q\ell^n q^{\lambda_{1n}} y;q)_{\lambda_{n1}}}{\prod_{j=1}^{n-1}(q^{\lambda_1-\lambda_{n-j+1}+1}\ell^{n-j};q)_{\lambda_{n-j+1}-\lambda_{n-j}}} \times \phi_D\left[\begin{matrix} a'; b'_1, \ldots, b'_{n-1} \\ c \end{matrix}; q; x_1, \ldots, x_{n-1}\right] \tag{5.18}$$

where $\lambda_{ij} \equiv \lambda_i - \lambda_j$ and

$$a' = y \qquad c = q\ell^n q^{\lambda_{1n}} y \qquad x_j = q\ell^{n-j} q^{\lambda_1 - \lambda_{n-j}} \qquad b'_j = q^{\lambda_{n-j} - \lambda_{n-j+1}} \tag{5.19}$$

for $j = 1, \ldots, n-1$. Another expression for $S_\lambda^{(\ell;q)}(y)$ reads

$$S_\lambda^{(\ell;q)}(y) = y^{\lambda_1} (q\ell y; q)_{(1-n)g} \left(\prod_{i=1}^{n-1} (a_i; q)_g \right) \phi_D \left[\begin{matrix} y; \ell^{-1}, \ldots, \ell^{-1} \\ q\ell y \end{matrix} ; q; a_1, \ldots, a_{n-1} \right]. \tag{5.20}$$

Proof. Equation (5.18) is obtained by substituting the parameters (5.19) in Andrews' formula (A.13) for the q-Lauricella function and comparing the result to (5.1). Note that $c/a' \equiv a_1$, $x_j \equiv a_{j+1}$, $b'_j x_j \equiv b_j$. Similarly, substituting in (A.13) the parameters $a' = y$, $c = q\ell y$, $b'_j = \ell^{-1}$, $x_j = a_j$ $(j = 1, \ldots, n-1)$ such that $c/a' \equiv a_n$, $b'_j x_j \equiv b_j$, one arrives at (5.20). □

Corollary 1. Substituting the definition (A.12) of ϕ_D in equation (5.18) we obtain another explicit representation for $S_\lambda^{(\ell;q)}(y)$:

$$\begin{aligned} S_\lambda^{(\ell;q)}(y) = y^{\lambda_1} & \frac{1}{\prod_{j=1}^{n-1} (q^{\lambda_1 - \lambda_{n-j+1} + 1} \ell^{n-j}; q)_{\lambda_{n-j+1} - \lambda_{n-j}}} \\ & \times \sum_{k_1=0}^{\lambda_n - \lambda_{n-1}} \cdots \sum_{k_{n-1}=0}^{\lambda_2 - \lambda_1} (q\ell^n q^{\lambda_{1n} + k_1 + \cdots + k_{n-1}} y; q)_{\lambda_{n1} - k_1 - \cdots - k_{n-1}} (y; q)_{k_1 + \cdots + k_{n-1}} \\ & \times \prod_{j=1}^{n-1} \frac{(q^{\lambda_{n-j} - \lambda_{n-j+1}}; q)_{k_j} (q\ell^{n-j} q^{\lambda_1 - \lambda_{n-j}})^{k_j}}{(q; q)_{k_j}} . \end{aligned} \tag{5.21}$$

Corollary 2. It is also possible to represent $S_\lambda^{(\ell;q)}(y)$ as a q-integral (A.8):

$$\begin{aligned} S_\lambda^{(\ell;q)}(q^x) = q^{\lambda_1 x} & \frac{(q\ell^n q^{\lambda_{1n}} q^x; q)_{\lambda_{n1}}}{\prod_{j=1}^{n-1} (q^{\lambda_1 - \lambda_{n-j+1} + 1} \ell^{n-j}; q)_{\lambda_{n-j+1} - \lambda_{n-j}}} \\ & \times \frac{1}{B_q(x, \lambda_{1n} + 1 - ng)} \int_0^1 \mathrm{d}_q t \; t^{x-1} \frac{(tq; q)_{\lambda_{1n} - ng}}{\prod_{j=1}^{n-1} (tq\ell^{n-j} q^{\lambda_1 - \lambda_{n-j}}; q)_{\lambda_{n-j} - \lambda_{n-j+1}}} . \end{aligned} \tag{5.22}$$

To obtain equation (5.22) rewrite Andrews' formula (A.13) as a q-integral

$$\phi_D \left[\begin{matrix} q^\alpha; q^{\beta_1}, \ldots, q^{\beta_{n-1}} \\ q^\gamma \end{matrix} ; q; x_1, \ldots, x_{n-1} \right] = \frac{1}{B_q(\alpha, \gamma - \alpha)} \int_0^1 \mathrm{d}_q t \; t^{\alpha-1} \frac{(tq; q)_{\gamma-\alpha-1}}{\prod_{j=1}^{n-1} (tx_j; q)_{\beta_j}} . \tag{5.23}$$

and substitute

$$\alpha = x \quad (y = q^x) \qquad \gamma = \lambda_{1n} + 1 - ng + x$$

$$\beta_j = \lambda_{n-j} - \lambda_{n-j+1} \qquad x_j = q\ell^{n-j} q^{\lambda_1 - \lambda_{n-j}} .$$

The rest of the results are concerned with the separated q-difference equations for the polynomials $S_\lambda^{(\ell;q)}(y)$.

Proposition 11. The polynomial $f(y) := S_\lambda^{(\ell;q)}(y)$ (5.1) satisfies the q-difference equation

$$\sum_{k=0}^{n}(-1)^k \ell^{[(n-1)/2]k}(1-q^k\ell^k y)(y;q)_k\,(q^{k+1}\ell^n y;q)_{n-k}\,h_{n-k}\,f(q^k y) = 0 \tag{5.24}$$

where the h_k are given by (3.9) and, as in the classical case (2.8), we assume $h_0 \equiv 1$.

Proof. Using the definitions (5.1) and (5.2), together with (3.9), it is a matter of straightforward calculation to transform the q-difference equation (A.10) for ${}_n\phi_{n-1}$ into (5.24). □

In fact, the factor $(1-q^n\ell^n y)$ can be cancelled from (5.24) which results in the formula

$$(-1)^n\ell^{n(n-1)/2}(y;q)_n h_0 f(q^n y)$$
$$+\sum_{k=0}^{n-1}(-1)^k \ell^{[(n-1)/2]k}(1-q^k\ell^k y)(y;q)_k\,(q^{k+1}\ell^n y;q)_{n-k-1}\,h_{n-k}\,f(q^k y) = 0\,. \tag{5.25}$$

In the $n=3$ case the q-difference equation (5.24) takes the form

$$\mathcal{D}(y, Y; h_1, h_2, h_3) f(y) = 0$$

where $\mathcal{D}$ is given by (4.11), or, explicitly,

$$(1-qy)(1-q^2y)\ell^3 f(q^3y) - (1-qy)(1-q^2\ell^2y)\ell^2 h_1 f(q^2y)$$
$$+(1-q\ell y)(1-q^2\ell^3 y)\ell h_2 f(qy) - (1-q\ell^3 y)(1-q^2\ell^3 y)h_3 f(y) = 0. \tag{5.26}$$

Proposition 12. Let

$$G_n^{(0)} := \mathbb{Z}\cup\tfrac{1}{2}\mathbb{Z}\cup\cdots\cup\frac{1}{n-1}\mathbb{Z} \qquad G_n^{(1)} := \left\{\frac{1}{n},\frac{2}{n},\ldots,\frac{n-2}{n}\right\}.$$

Then, for all $g > 0$ except for the finite number of points $g \in G_n \equiv G_n^{(0)} \cap G_n^{(1)}$, the separated polynomial $f(y) := S_\lambda^{(\ell;q)}(y)$, equation (5.1), is the only, up to a constant factor, Laurent-polynomial solution to the q-difference equation (5.24).

In particular, $G_3 = \emptyset$, so for $n = 3$ the uniqueness of the Laurent-polynomial (LP) solution holds $\forall g > 0$.

Proof. In the non-relativistic case [2] the analogue of equation (5.24) is a differential equation having three regular singularities: $0, 1, \infty$, and the uniqueness of the LP solution is proved by analysis of the corresponding characteristic exponents. As shown below, the argument can be translated rather directly to the q-difference case.

Let $f(y)$ be a non-zero Laurent-polynomial solution to (5.24) or, equivalently, (5.25). Then, subsituting into (5.25) the values $y = q^{-j}$, $j = 0, 1, 2, \ldots$, one observes that $f(q^{-j})$ can be determined recursively, starting from $f(1)$ since the factor $(y;q)_k$ cuts away the terms with $k > j$. The only obstacle could be the vanishing of the factor $(q^{k+1}\ell^n y;q)_{n-k-1}$ for $k = 0$ which may happen only for $g \in G_n^{(1)}$. Suppose $g \notin G_n^{(1)}$. Then it is sufficient to use the fact that any Laurent polynomial vanishing on a countable set vanishes identically. It follows that, first, $f(1) \neq 0$ for any non-zero LP solution and, second, any two non-zero LP solutions are proportional, in particular to the standard solution $S_\lambda^{(\ell;q)}(y)$.

Instead of the sequence $y = q^{-j}$ one can take $y = q^j\ell^{-n}$ and use the same argument. Note that the above recursive process is the exact analogue of the Taylor series expansion about $y = 1$ in the non-relativistic case.

On the other hand, a similar argument works with expansion about 0 or ∞. Substituting in (5.24) the expansion $f(y) = \sum_{k=k_-}^{k_+} f_k y^k$ one obtains the $(n+2)$-term recurrence relation $\sum_{j=0}^{n+1} A_{kj} f_{k-j} = 0$ for f_k. The 'boundary' coefficients A_{k0} and $A_{k,n+1}$ have the simple form

$$A_{k0} = (-1)^n \ell^{n(n-1)/2} \prod_{j=1}^{n} (q^k - q^{\lambda_j} \ell^{1-j}) \tag{5.27a}$$

$$A_{k,n+1} = -\left(\frac{\ell}{q}\right)^{n(n+1)/2} \prod_{j=1}^{n} (q^k - q^{\lambda_j+n+1} \ell^{n-j}). \tag{5.27b}$$

Suppose $g \notin G_n^{(0)}$. Then, since $\ell = q^{-g}$, the coefficient A_{k0} vanishes only for $k = \lambda_1$, and $A_{k,n+1} = 0$ only for $k = \lambda_n + n + 1$. Hence inevitably $k_- = \lambda_1$, $k_+ = \lambda_n$, and the coefficients f_k are determined recursively in a unique way starting from f_{λ_1} or f_{λ_n} which proves the uniqueness of the LP solution. □

The question as to whether the uniqueness of the LP solution really breaks for $g \in G_n$, still remains open.

It would be interesting to strengthen the above result.

Conjecture 2. The equation (5.24) with free parameters h_j has a polynomial solution only for h_j given by (3.9) and $\lambda = \{\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n\} \in \mathbb{Z}^n$.

6. Discussion

The results of the present paper generalize those of [2] obtained for the Calogero–Sutherland model and Jack polynomials to the case of the Ruijsenaars model and Macdonald polynomials. In the non-relativistic limit $\hbar \to 0$, $g =$ constant, the Hamiltonians H_k, operator M, separated polynomials S and equations for them go over into the corresponding objects described in [2].

The crucial element of our approach is the operator identity (4.13) which allows us to prove the factorization (4.3) of $MP_\lambda^{(\ell;q)}$ and thus to establish the separation of variables. The identity (4.13) is apparently a quantum analogue of the characteristic equation for the classical Lax operator. Moreover, the kernel $\mathcal{M}$ can be considered as a collection of eigenfunctions to the quantized separation variables y_j describing thus the change of basis from 't-representation' to 'y-representation'. Although these analogies with the classical inverse scattering method proved to be useful as a heuristic tool for finding SoV for quantum integrable systems [1], their algebraic/geometric origin is yet to be cleared up.

An interesting problem is to search for alternative forms of M. We have presented two descriptions of M here: an analytical one, equation (4.1), in terms of the Askey–Wilson integral, and an algebraic one, equation (4.31), in terms of the basis $p_{jk\nu}$. Our study of M is based mainly on the analytical definition. It would be interesting also to develop the theory of M based entirely on the algebraic definition, and, in particular, to give a purely algebraic proof of the identity (4.13).

When our work was close to being finished we became aware of the preprint [17] of Mangazeev addressing the same problem of SoV for A_2 Macdonald polynomials. His proposal for the operator M is different from ours, using a q-integral rather than a contour integral as we do. Some of his arguments are quite formal: for instance, the expressions with the ${}_6\psi_6$-series he uses as a final result are divergent. It seems that our choice of M, compared with that of [17], allows one to overcome the problems of convergence of the

q-integral and to obtain explicit expressions for M^{-1} and action of M on polynomials. Still, the problem of representing M as a q-integral seems to deserve further consideration.

Although we can predict the form of the separation polynomial $S_\lambda^{(\ell;q)}(y)$ for the n-particle case and have studied it in detail (section 5), the corresponding n-particle generalization of the kernel $\mathcal{M}$ is not yet clear, so it remains an open problem to separate variables for the A_{n-1} Macdonald polynomials for $n > 3$.

In fact, there are infinitely many 'separating' operators $M^{(n)}$, since for any choice of c_λ the operator defined as

$$M^{(n)} : P_\lambda^{(\ell;q)}(t_1, \ldots, t_n) \to c_\lambda x^{|\lambda|} \prod_{j=1}^{n-1} S_\lambda^{(\ell;q)}(y_j) \tag{6.1}$$

will serve the purpose. The genuine problem, however, is to choose the coefficients c_λ in such a way that the corresponding kernel $\mathcal{M}^{(n)}$ is given by an explicit expression that generalizes (4.2).

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Appendix A. Formulae from q-analysis

For the reader's convenience, we have collected here the most important definitions and formulae from q-analysis used in the main body of the paper. For references see [10–13]. Especially useful for practical calculations is the collection of formulae in [10, appendices I and II]. Throughout the text it is assumed that $0 < q < 1$.

The q-shifted factorial and its generalizations are defined as

$$(a;q)_0 := 1 \qquad (a;q)_k := (1-a)(1-aq)\cdots(1-aq^{k-1}) \qquad k = 1, 2, \ldots \tag{A.1}$$

$$(a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k) \qquad (x;q)_\alpha = \frac{(x;q)_\infty}{(q^\alpha x;q)_\infty} \quad \alpha \in \mathbb{C} \tag{A.2}$$

$$(a_1, a_2, \cdots, a_n; q)_k := (a_1;q)_k (a_2;q)_k \cdots (a_n;q)_k \qquad k = 0, 1, 2, \ldots \text{ or } \infty. \tag{A.3}$$

Note the useful relations

$$(qx;q)_\alpha = \frac{1-q^\alpha x}{1-x}(x;q)_\alpha \qquad (q^{-1}x;q)_\alpha = \frac{1-q^{-1}x}{1-q^{\alpha-1}x}(x;q)_\alpha. \tag{A.4}$$

We also make use of the q-binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}} \qquad k = 0, 1, \ldots, n \tag{A.5}$$

the q-gamma and q-beta functions

$$\Gamma_q(z) = \frac{(q;q)_\infty}{(q^z;q)_\infty(1-q)^{z-1}} \qquad \Gamma_q(z+1) = \frac{1-q^z}{1-q}\Gamma_q(z) \tag{A.6}$$

$$B_q(a,b) = \frac{\Gamma_q(a)\Gamma_q(b)}{\Gamma_q(a+b)} = (1-q)\frac{(q,q^{a+b};q)_\infty}{(q^a,q^b;q)_\infty} \tag{A.7}$$

the q-integral

$$\int_0^1 \mathrm{d}_q t\; f(t) := (1-q)\sum_{k=0}^\infty f(q^k)q^k \tag{A.8}$$

and the basic hypergeometric series ($n \in \mathbb{Z}_{\geqslant 0}$)

$${}_n\phi_{n-1}\left[\begin{matrix} a_1,\ldots,a_n \\ b_1,\ldots,b_{n-1}\end{matrix};q,y\right] := \sum_{k=0}^\infty \frac{(a_1,\ldots,a_n;q)_k}{(q,b_1,\ldots,b_{n-1};q)_k} y^k \qquad |y|<1. \tag{A.9}$$

Denoting expression (A.9) by $f(y)$, we observe that it satisfies the nth order q-difference equation, see [14] and [12, section 2.12.3]:

$$\left\{ y\prod_{k=1}^n (1-a_kY) - \prod_{k=1}^n (1-q^{-1}b_kY)\right\} f(y) = 0 \tag{A.10}$$

where $(Yf)(y) := f(qy)$ and $b_n \equiv q$.

The summation formula for the ${}_1\phi_0$ (q-binomial series) is

$${}_1\phi_0\left[\begin{matrix} a \\ - \end{matrix};q,y\right] \equiv \sum_{k=0}^\infty \frac{(a;q)_k}{(q;q)_k} y^k = \frac{(ay;q)_\infty}{(y;q)_\infty} \qquad |y|<1. \tag{A.11}$$

The ϕ_D-type q-Lauricella function [13, 15] of the $n-1$ variables x_j is a multi-variable generalization of the basic hypergeometric series:

$$\phi_D\left[\begin{matrix} a';b'_1,\ldots,b'_{n-1} \\ c\end{matrix};q;x_1,\ldots,x_{n-1}\right] := \sum_{k_1,\ldots,k_{n-1}=0}^\infty \frac{(a';q)_{k_1+\cdots+k_{n-1}}}{(c;q)_{k_1+\cdots+k_{n-1}}} \prod_{j=1}^{n-1} \frac{(b'_j;q)_{k_j} x_j^{k_j}}{(q;q)_{k_j}}. \tag{A.12}$$

Andrews [16] has found that ϕ_D can be expressed in terms of the basic hypergeometric function ${}_n\phi_{n-1}$ of one variable:

$$\phi_D\left[\begin{matrix} a';b'_1,\ldots,b'_{n-1} \\ c\end{matrix};q;x_1,\ldots,x_{n-1}\right] = \frac{(a',b'_1x_1,\ldots,b'_{n-1}x_{n-1};q)_\infty}{(c,x_1,\ldots,x_{n-1};q)_\infty}$$

$$\times {}_n\phi_{n-1}\left[\begin{matrix} c/a', & x_1, & \cdots, & x_{n-1} \\ & b'_1x_1, & \ldots, & b'_{n-1}x_{n-1}\end{matrix};q,a'\right]. \tag{A.13}$$

Our main technical tool, on which the proof of the main theorem (theorem 3) depends, is the famous Askey–Wilson integral ([10, section 6.1], [11, section 2.6]). Let

$$w(a,b,c,d;t) := \frac{(t^2,t^{-2};q)_\infty}{(at,at^{-1},bt,bt^{-1},ct,ct^{-1},dt,dt^{-1};q)_\infty}. \tag{A.14}$$

Then

$$\frac{1}{2\pi \mathrm{i}}\int_{\Gamma_{abcd}} \frac{\mathrm{d}t}{t}\, w(a,b,c,d;t) = \frac{2(abcd;q)_\infty}{(q,ab,ac,ad,bc,bd,cd;q)_\infty}. \tag{A.15}$$

The cycle Γ_{abcd} depends on the parameters a, b, c, d and is defined as follows. Let $C_{z,r}$ be the anticlockwise-oriented circle with centre z and radius r.

If $|a|, |b|, |c|, |d| < 1$ then $\Gamma_{abcd} = C_{0,1}$. The identity (A.15) can be continued analytically for values of the parameters a, b, c, d outside the unit circle provided the cycle Γ_{abcd} is deformed appropriately. In the general case

$$\Gamma_{abcd} = C_{0,1} + \sum_{x=a,b,c,d} \sum_{\substack{k\geqslant 0 \\ |x|q^k>1}} (C_{xq^k,\varepsilon} - C_{x^{-1}q^{-k},\varepsilon}) \tag{A.16}$$

where ε is small enough for $C_{x^{\pm 1}q^{\pm k},\varepsilon}$ to encircle only one pole of the denominator.

The following formulae are useful when studying the classical and non-relativistic limits of the quantum Ruijsenaars model. Both correspond to $\hbar \to 0$, $q = \mathrm{e}^{-\hbar} \to 1$ and differ only in the behaviour of ℓ (3.5). As $q \uparrow 1$

$$(x; q)_\alpha \to (1-x)^\alpha \tag{A.17}$$

$$\frac{(q^\alpha; q)_k}{(1-q)^k} \to (\alpha)_k := \alpha(\alpha+1)\cdots(\alpha+k-1) \tag{A.18}$$

$${}_n\phi_{n-1}\left[\begin{matrix} q^{\alpha_1}, \ldots, q^{\alpha_n} \\ q^{\beta_1}, \ldots, q^{\beta_{n-1}} \end{matrix}; q, y\right] \to {}_nF_{n-1}\left[\begin{matrix} \alpha_1, \ldots, \alpha_n \\ \beta_1, \ldots, \beta_{n-1} \end{matrix}; y\right] \tag{A.19}$$

where ${}_nF_{n-1}$ is the standard (generalized) hypergeometric series, and finally (see [9, section 2.5, corollary 10]):

$$\ln(x; q)_\infty = -\hbar^{-1}\,\mathrm{Li}_2(x) + \tfrac{1}{2}\ln(1-x) + \mathrm{O}(\hbar) \qquad x \in (0, 1). \tag{A.20}$$

Appendix B. The operator $M_{\alpha\beta}$

In this section the results concerning the two-parametric generalization $M_{\alpha\beta}$ of the one-parametric operator family $M \equiv M_{g,2g}$ studied in the main text are collected. It is an open question whether $M_{\alpha\beta}$ provides a SoV for some integrable model.

Let us substitute in the Askey–Wilson integral weight $w(a, b, c, d; t)$, equation (A.14), the values

$$a = sq^{\alpha/2} \qquad b = s^{-1}q^{\alpha/2} \qquad c = rq^{\beta/2} \qquad d = r^{-1}q^{\beta/2} \tag{B.1}$$

and introduce the notation (quantum analogue of (2.30))

$$\mathcal{L}_q(\nu; x, y) := (\nu xy, \nu xy^{-1}, \nu x^{-1}y, \nu x^{-1}y^{-1}; q)_\infty. \tag{B.2}$$

The kernel

$$\mathcal{K}_{\alpha\beta}(r, s \mid t) := w(sq^{\alpha/2}, s^{-1}q^{\alpha/2}, rq^{\beta/2}, r^{-1}q^{\beta/2}; t) = \frac{(t^2, t^{-2}; q)_\infty}{\mathcal{L}_q\big(q^{\alpha/2}; s, t\big)\,\mathcal{L}_q\big(q^{\beta/2}; r, t\big)} \tag{B.3}$$

defines the integral operator

$$K_{\alpha\beta} : f(t) \to \frac{1}{2\pi\mathrm{i}} \int_{\Gamma^{rs}_{\alpha\beta}} \frac{\mathrm{d}t}{t}\,\mathcal{K}_{\alpha\beta}(r, s \mid t) f(t) \tag{B.4}$$

the contour $\Gamma^{rs}_{\alpha\beta}$ being obtained from Γ_{abcd}, equation (A.16), by the substitutions (B.1).

Using the Askey–Wilson integral (A.15) we then obtain the formula

$$K_{\alpha\beta} : 1 \to \frac{2B_q(\alpha, \beta)}{(1-q)(q;q)^2_\infty\,\mathcal{L}_q\big(q^{(\alpha+\beta)/2}; r, s\big)}. \tag{B.5}$$

Now we introduce the operator $M_{\alpha\beta}$

$$M_{\alpha\beta} = (K_{\alpha\beta} \cdot 1)^{-1} \circ K_{\alpha\beta} \tag{B.6}$$

so that $M : 1 \to 1$, having the kernel

$$\mathcal{M}_{\alpha\beta}(r, s \mid t) = \frac{(1-q)(q;q)_\infty^2 (t^2, t^{-2}; q)_\infty \, \mathcal{L}_q\big(q^{(\alpha+\beta)/2}; r, s\big)}{2B_q(\alpha,\beta)\, \mathcal{L}_q\big(q^{\alpha/2}; s, t\big)\, \mathcal{L}_q\big(q^{\beta/2}; r, t\big)}. \tag{B.7}$$

The kernel $\mathcal{M}$ (4.2) studied in section 4 is obtained from $\mathcal{M}_{\alpha\beta}$ (B.7) after the substitutions

$$\begin{aligned} &\alpha = g \qquad \beta = 2g \qquad q^{-g} = \ell \\ &r = t_+ = y_+ \ell^{\frac{3}{2}} \qquad s = y_- \qquad t = t_-. \end{aligned} \tag{B.8}$$

It is natural to think of $M_{\alpha\beta}$ as acting on the space $\mathrm{Ref}(t)$ of the reflexive (invariant w.r.t. $t \to t^{-1}$) Laurent polynomials in t. Consider a Laurent polynomial $R^{\alpha\beta}_{j_1 j_2 k_1 k_2} \in \mathrm{Ref}(t)$, $j_{1,2}, k_{1,2} \in \mathbb{Z}_{\geqslant 0}$

$$\begin{aligned} R^{\alpha\beta}_{j_1 j_2 k_1 k_2}(t) := {}& (q^{\alpha/2}st, q^{\alpha/2}st^{-1}; q)_{j_1} (q^{\alpha/2}s^{-1}t, q^{\alpha/2}s^{-1}t^{-1}; q)_{j_2} \\ & \times (q^{\beta/2}rt, q^{\beta/2}rt^{-1}; q)_{k_1} (q^{\beta/2}r^{-1}t, q^{\beta/2}r^{-1}t^{-1}; q)_{k_2}. \end{aligned} \tag{B.9}$$

Using the obvious identity

$$\mathcal{K}_{\alpha\beta}(r, s \mid t) R^{\alpha\beta}_{j_1 j_2 k_1 k_2}(t) = \mathcal{K}_{\alpha+j_1+j_2, \beta+k_1+k_2}\left(rq^{(k_1-k_2)/2}, sq^{(j_1-j_2)/2} \;\middle|\; t \right) \tag{B.10}$$

together with (B.5), we obtain the formula for the action of $M_{\alpha\beta}$ on the Laurent polynomials

$$\begin{aligned} M_{\alpha\beta} : R^{\alpha\beta}_{j_1 j_2 k_1 k_2} \to {}& \frac{(q^\alpha; q)_{j_1+j_2} (q^\beta; q)_{k_1+k_2}}{(q^{\alpha+\beta}; q)_{j_1+j_2+k_1+k_2}} (q^{(\alpha+\beta)/2} rs; q)_{j_1+k_1} (q^{(\alpha+\beta)/2} rs^{-1}; q)_{j_2+k_1} \\ & \times (q^{(\alpha+\beta)/2} r^{-1}s; q)_{j_1+k_2} (q^{(\alpha+\beta)/2} r^{-1}s^{-1}; q)_{j_2+k_2}. \end{aligned} \tag{B.11}$$

The set of polynomials $R^{\alpha\beta}_{j_1 j_2 k_1 k_2}$ is rich enough to choose from it a basis in $\mathrm{Ref}(t)$, for instance

$$p^\beta_\nu(t) := R^{\alpha\beta}_{00\nu 0}(t) \equiv (q^{\beta/2}rt, q^{\beta/2}rt^{-1})_\nu \qquad \nu = 0, 1, 2, \ldots. \tag{B.12}$$

More correctly, since $p^\beta_\nu(t) = (-1)^\nu q^{\nu(\nu-1+\beta)/2} r^\nu (t^\nu + t^{-\nu}) +$ lower-order terms, $p^\beta_\nu(t)$ is a basis in the space of reflexive Laurent polynomials in the variable t with coefficients from $\mathrm{Ref}(r)$. The specialization of equation (B.11)

$$M_{\alpha\beta} : p^\beta_\nu(t) \to \frac{(q^\beta; q)_\nu}{(q^{\alpha+\beta}; q)_\nu} p^{\alpha+\beta}_\nu(s) \tag{B.13}$$

thus provides a tool for explicitly calculating the action of M on any polynomial $\in$ Ref(t).

Analysing equation (B.13) one obtains the following statement.

Proposition 13. Let $f \in \mathrm{Ref}(t)$ and suppose f has the parity σ that is $f(-t) = (-1)^\sigma f(t)$. Let $M_{\alpha\beta} : f \to F$. Then $F \in \mathrm{Ref}(r) \otimes \mathrm{Ref}(s)$, $F(r,s) = F(s,r)|_{\alpha\leftrightarrow\beta}$, $F(-r,-s) = (-1)^\sigma F(r,s)$.

The integral operator $M_{\alpha\beta}$ simplifies drastically when one of the parameters α, β takes a negative integer value.

Proposition 14. Let $\alpha \in \mathbb{Z}_{\leqslant 0}$. Then $M_{\alpha\beta}$ turns into the q-difference operator of order $-\alpha$:

$$M_{\alpha\beta} : f(t) \to \sum_{k=0}^{-\alpha} \xi_k(r, s) f(q^{k+\alpha/2} s) \tag{B.14}$$

where

$$\xi_k(r, s) = (-1)^k q^{-k(k-1)/2} \begin{bmatrix} -\alpha \\ k \end{bmatrix}_q s^{-2k}(1 - q^{-\alpha-2k} s^{-2})$$
$$\times \frac{(q^{(\alpha+\beta)/2} rs, q^{(\alpha+\beta)/2} r^{-1} s; q)_k (q^{(\alpha+\beta)/2} rs^{-1}, q^{(\alpha+\beta)/2} r^{-1} s^{-1}; q)_{-\alpha-k}}{(q^{\alpha+\beta}; q)_{-\alpha} (q^{-k} s^{-2}; q)_{1-\alpha}}. \tag{B.15}$$

Proof. Instead of analysing the degeneration of the integral operator defined by the kernel (B.7) it is easier to study the action of $M_{\alpha\beta}$ on the basic polynomials $p_\nu^\beta(t)$, equation (B.12).

Substituting $f(t) := p_\nu^\beta(t)$ in (B.14) and using (B.13) we obtain, after simplification, the equality

$$\sum_{k=0}^{-\alpha} \xi_k(r, s) \frac{(q^{(\alpha+\beta)/2+\nu} rs; q)_k (q^{(\alpha+\beta)/2+\nu} rs^{-1}; q)_{-\alpha-k}}{(q^{(\alpha+\beta)/2} rs; q)_k (q^{(\alpha+\beta)/2} rs^{-1}; q)_{-\alpha-k}} = \frac{(q^{\alpha+\beta+\nu}; q)_{-\alpha}}{(q^{\alpha+\beta}; q)_{-\alpha}} \tag{B.16}$$

which, after substituting (B.15) and making a series of elementary transformations (see [10, appendix I]), can be put into the form

$$\sum_{k=0}^{-\alpha} q^{(1-\alpha-\beta-\nu)k} \frac{(q^\alpha, q^{(\alpha+\beta)/2+\nu} rs, q^{(\alpha+\beta)/2} r^{-1} s, q^\alpha s^2, q^{1+\alpha/2} s, -q^{1+\alpha/2} s; q)_k}{(q, q^{(\alpha-\beta)/2+1} rs, q^{(\alpha-\beta)/2+1-\nu} r^{-1} s, qs^2, q^{\alpha/2} s, -q^{\alpha/2} s; q)_k}$$
$$= \frac{(q^{1-\beta-\nu}, q^{1+\alpha} s^2; q)_{-\alpha}}{(q^{(\alpha-\beta)/2+1} rs, q^{(\alpha-\beta)/2+1-\nu} r^{-1} s; q)_{-\alpha}} \tag{B.17}$$

identical to the summation formula [10, equation (II.21)]:

$${}_6\phi_5 \left[\begin{matrix} a, qa^{\frac12}, -qa^{\frac12}, b, c, q^\alpha \\ a^{\frac12}, -a^{\frac12}, aq/b, aq/c, aq^{1-\alpha} \end{matrix} ; q, \frac{aq^{1-\alpha}}{bc} \right] = \frac{(aq, aq/bc; q)_{-\alpha}}{(aq/b, aq/c; q)_{-\alpha}} \tag{B.18}$$

for the following identification of the parameters

$$a = q^\alpha s^2 \qquad b = q^{(\alpha+\beta)/2} r^{-1} s \qquad c = q^{(\alpha+\beta)/2+\nu} rs. \tag{B.19}$$

By the symmetry $\alpha \leftrightarrow \beta$, $r \leftrightarrow s$ the corresponding statement can be proved for $\beta \in \mathbb{Z}_{\leqslant 0}$. □

To determine the inversion of $M_{\alpha\beta}$ let us think of r as a parameter and of $M_{\alpha\beta}$ as an operator $M_{\alpha\beta}^r : \mathrm{Ref}(t) \to \mathrm{Ref}(s) : f(t) \to F(s)$. Then, applying $M_{\alpha\beta}^r$ to the basis $p_\nu^\beta(t)$, equation (B.12), and inverting (B.13), we obtain the following statement.

Proposition 15. The inversion formula for $M_{\alpha\beta}^r$ is

$$\left(M_{\alpha\beta}^r\right)^{-1} = M_{-\alpha,\alpha+\beta}^r. \tag{B.20}$$

The corresponding kernel is

$$\widetilde{\mathcal{M}}_{\alpha\beta}^r(t \mid s) = \frac{(1-q)(q;q)_\infty^2 (s^2, s^{-2}; q)_\infty \, \mathcal{L}_q\left(q^{\beta/2}; r, t\right)}{2B_q(-\alpha, \alpha+\beta)\, \mathcal{L}_q\left(q^{-\alpha/2}; s, t\right) \mathcal{L}_q\left(q^{(\alpha+\beta)/2}; r, s\right)}. \tag{B.21}$$

References

[1] Sklyanin E K 1995 Separation of variables. New trends *Prog. Theor. Phys. Suppl.* **118** 35–60

[2] Kuznetsov V B and Sklyanin E K 1995 Separation of variables in A_2 type Jack polynomials *RIMS Kokyuroku* **919** 27; 1995 *Preprint* solv-int/9508002

[3] Sklyanin E K 1995 Dynamical r-matrices for the elliptic Calogero–Moser model *St Petersburg Math. J.* **6** 397–406

Braden H W and Suzuki T 1994 R-matrices for elliptic Calogero–Moser models *Lett. Math. Phys.* **30** 147–58

[4] Ruijsenaars S N M 1987 Complete integrability of relativistic Calogero–Moser systems and elliptic function identities *Commun. Math. Phys.* **110** 191–213

[5] Macdonald I G 1979 *Symmetric Functions and Hall Polynomials* (Oxford: Oxford University Press) (2nd edn 1995)

[6] Macdonald I G 1990 Orthogonal polynomials associated with root systems *Orthogonal Polynomials: Theory and Practice (Proc. NATO ASI on Orthogonal Polynomials and Their Applications (Colombus, OH))* (NATO ASI Series C **294**) ed P Nevai (Dordrecht: Kluwer)

[7] van Diejen J F 1995 Commuting difference operators with polynomial eigenfunctions *Compositio Math.* **95** 183–223

[8] Arnold V I 1989 *Mathematical Methods of Classical Mechanics* 2nd edn (Berlin: Springer)

[9] Kirillov A N 1995 Dilogarithm identities *Progr. Theor. Phys. Suppl.* **118** 61–142

[10] Gasper G and Rahman M 1990 *Basic Hypergeometric Series* (Cambridge: Cambridge University Press)

[11] Koornwinder T H 1994 Compact quantum qroups and q-special functions *Representations of Lie Groups and Quantum Qroups* (Pitman Research Notes in Mathematics **311**) ed V Baldoni and M A Picardello (London: Longman) pp 46–128

[12] Exton H 1983 *q-Hypergeometric Functions and Applications* (New York: Wiley)

[13] Erdelyi A *et al* 1953 *Higher Transcendental Functions* (New York: McGraw-Hill)

[14] Jackson F H 1910 q-difference equations *Am. J. Math.* **32** 305–14

[15] Noumi M 1992 Quantum grassmannians and q-hypergeometric series *CWI Q.* **5** 293–307

Floreanini R, Lapointe L and Vinet L 1994 A quantum algebra approach to basic multivariable special functions *J. Phys. A: Math. Gen.* **27** 6781–97

[16] Andrews G 1972 Summations and transformations for basic Appell series *J. Lond. Math. Soc. (2)* **4** 618–22

[17] Mangazeev V V 1995 A_2 Macdonald polynomials: a separation of variables *Preprint* q-alg/9512003